

# Inventory Management with Purchase Order Errors and Rework

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## Abstract

In a retail distribution center, the fulfillment of a purchase order (PO) occasionally arrives with errors that the retailer can correct through rework. PO fulfillment errors, which we will refer to as PO errors in brevity henceforth, include “ticket errors,” in which an incorrect stock keeping unit (SKU) number or sale price is put on the tickets attached to some items in a PO. These errors are different from the random yield errors well-known in the literature. Using data from a large retail chain, we determine key properties of these errors, in particular how they vary with order complexity. We define order complexity as having two components, namely average order quantity per SKU and number of SKUs on a PO. In order to study their influence on a retailer’s order policy and inventory system cost, we propose one deterministic and one stochastic  $(Q, R)$  inventory model for POs with a single SKU that account for these correctable errors. We study the deterministic inventory model analytically to gain some insights, and conduct numerical experiments to analyze the stochastic model. Our parameters for the numerical experiments are estimated from real retail data. In order to help retailers understand and quantify the cost impact of PO errors and rework, we compare the performance of our adjusted order policy with standard policies that ignore correctable PO errors. In addition, we propose procedures and provide qualitative guidance for a retailer to easily identify SKUs that are more prone to these errors. Retailers can then adjust their inventory models to account for these PO errors and work with their vendors to reduce the error incidence for these high impact SKUs.

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# 1 Introduction

Retailers replenish inventory through their supply networks, which are composed of various vendors, retail distribution centers (DCs), and retail stores. Products ordered by retailers are usually shipped from vendors to retail DCs, and then distributed to retail stores. Inventory management at the retail DCs is complicated and challenging not only because of the large variety of products and long lead time, but also due to vendor errors. A vendor error occurs when a vendor does not deliver to the DC as specified by the order contract. Such vendor errors incur additional labor cost and complicate retail operations, and are termed vendor non-compliance problems in the business literature.

Retailers penalize vendor non-compliance in the form of chargebacks. Chargebacks are a significant cost for most vendors. From 4% to 10% of all open items on accounts receivable were affected by chargeback deductions, according to the Credit Research Foundation. Chargebacks typically cut off 2% to 10% of a vendor's overall revenue, according to the National Chargebacks Management Group (NCMG) of Charlotte, North Carolina (Zieger (2003)). Chargebacks also reflect, to some extent, the considerable cost incurred by retailers as a result of vendor non-compliance. Retailers use chargebacks as a means to recover their additional cost incurred as a result of vendor non-compliance problems (Aron (1998), and Weber et al. (2005)). Chargebacks have been an issue of great contention between retailers and their vendors. While retailers do not consider chargebacks as a profit center, many vendors do (Atkinson (2007), and Aron (1998)). Some vendors hire a special credit consultancy to validate retailer chargebacks (Atkinson (2007)). Some vendors even file law suits to refuse chargeback payments that they feel unjustified. About 10% of vendors successfully challenge a chargeback in a given year (Zieger (2003)). Chargebacks, a product of vendor non-compliance problems, have added unnecessary cost for both retailers and their vendors and made the inventory supply process inefficient. Therefore, it is important for vendors, as well as retailers, to reduce vendor non-compliance problems. Ideally, retailers should work with their vendors to completely eliminate non-compliance problems. However, in practice, complete elimination may not be feasible. In this case, retailers can adjust their inventory policies to account for vendor non-compliance. We focus on how retailers should adjust their inventory policies in the presence of vendor non-compliance problems.

For a large retail chain called Omega<sup>1</sup>, the source of our data for this study, around 5% of its purchase orders (POs) (about 4,000 POs in a year) have vendor non-compliance problems. Under vendor non-compliance we include a range of PO fulfillment errors, such as quantity shortage errors and ticket errors. We will henceforth refer to PO fulfillment errors as PO errors in brevity. A "ticket error" occurs when the vendor of a PO put an incorrect stock keeping unit (SKU) number or sale price on some items in the PO. Whenever the fulfillment of a PO arrives at a retail DC with errors, the retailer needs to perform rework in order to deal with these errors. As a result, the

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<sup>1</sup>We disguise the real name at the request of the retailer.

retailer incurs additional labor cost, lead time, and lead time variability. The additional lead time is composed of two parts, namely the actual rework time and the waiting time at the retail DC. Due to the limited processing capacity at a retail DC, a PO might have to wait in the processing queue for all its previous POs to finish rework. For our partner retailer, the rework time of a PO is about 1.6 hours on average with a maximum of 3.6 days, and the waiting time is typically 2 to 3 days. In summary, PO errors add cost and complicate retail inventory management. Hence, our partner retailer penalizes its manufacturers millions of dollars for these errors in the form of vendor chargebacks.

Although one type of PO errors, random yield, has been extensively studied in the literature, there is to the best of our knowledge little research concerning other types of PO errors in the retail setting. Retailers have long realized the importance of dealing with these PO errors, considering the large amount of chargebacks they impose on their vendors. However, they do not know how to take these PO errors into account in their inventory policy, and have no idea of the real cost of these PO errors. Our work helps retailers understand the influence of PO errors on their inventory policy and cost. The objectives of our research are to determine (1) “PO error cost”, which is defined as the additional cost a retailer incurs due to PO errors, (2) the retailer’s new optimal order policy in anticipation of PO errors that lead to rework, and (3) “potential cost savings”, which is defined as the amount the retailer can save by adjusting its order policies to account for PO errors. In addition, in retail practice, it is useful for a retailer to easily identify SKUs with high potential cost savings without adjusting the existing inventory management system to account for PO errors. A retailer may need to manage hundreds of thousands of SKUs. If the retailer has purchased inventory management software from a specialized software vendor, there will be an additional cost to adjust and fine-tune the existing inventory system to account for PO errors for each SKU chosen. Under the emerging business model software-as-a-service, inventory management software vendors like Predictix<sup>2</sup> charge retailers per SKU to adjust the standard inventory system. Hence, the retailer may prefer to do this only for SKUs with high potential cost savings, using only readily available information (such as inventory cost parameters and demand estimates) to select such SKUs. It is also useful for a retailer to easily identify SKUs for which PO errors are particularly costly without adjusting the existing inventory management system to account for such errors. The retailer could then collaborate with its vendors to reduce error probability and magnitude for those SKUs. Since the retailer may not want to pay its inventory software vendor to calculate the cost of PO errors for each SKU, it needs a procedure to choose such SKUs based on properties that are easy to obtain. Our research endeavors to provide such procedures as well as some qualitative guidance for a retailer to easily identify SKUs with high potential cost savings or PO error cost.

We consider retail inventory management of a single item under continuous review in the presence of PO errors that are eligible for rework at the retail DC. Using data from our partner retailer,

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<sup>2</sup>Predictix is a retail consulting firm headquartered in Atlanta, GA. It serves companies in the retail, wholesale and consumer packaged goods industries. The service areas include inventory planning and allocation, assortment planning and shelf space allocation, pricing and promotions planning, as well as demand forecasting and inventory replenishment.

we have obtained the properties of these errors, in particular how PO errors vary with order complexity. We define order complexity as having two components, namely average order quantity per SKU and number of SKUs on a PO. Specifically, we find that incidence and magnitude of PO errors increase in both the average order quantity of SKUs and the number of SKUs on this PO. The properties of these correctable PO errors are different from standard assumptions in the random yield literature. We also find that the rework time for a PO increases in the error size of the PO. With these new properties of PO errors, we first study a deterministic inventory model analytically to obtain insights about the influence of PO errors. We show that (a)the optimal order quantity when accounting for PO errors is no less than the standard optimal order quantity. Both the potential cost savings and PO error cost (b)decrease with the fixed ordering cost, and (c)increase in the holding and backlogging cost as well as in the demand rate. These properties provide qualitative guidance for a retailer to identify SKUs with high potential cost savings or PO error cost. We then apply the single-item continuous time stochastic inventory model under a  $(Q, R)$  policy with random yield and rework to reflect the properties of PO errors observed in the empirical data analysis. Due to the complexity of the stochastic model, we conduct numerical experiments to analyze the influence of PO errors. The parameters for the numerical experiments are estimated using a proprietary data set from our partner retailer. Insights similar to those derived from the deterministic model are also observed in the numerical experiments. In addition, we find that PO errors cost our partner retailer tens of to hundreds of dollars per transaction (i.e., per PO), and that whether our partner retailer under or over charges its vendors for these PO errors depends on the inventory cost parameters (including the fixed ordering cost, the holding cost and the backlogging penalty cost).<sup>3</sup> Currently, our partner retailer penalizes PO errors by imposing a two-part charge on its vendors. A fixed amount is charged whenever there is an error in a PO, as well as a variable amount proportional to the rework time for that PO. The fixed amount does not fully capture the increased inventory cost due to PO errors when the holding cost is high. We also observe substantial potential cost savings that retailers can obtain when accounting for these PO errors in their inventory policies. Predictive models of potential cost savings and PO error cost, which are essential components of the procedures to easily identify SKUs with higher potential cost savings or PO error cost, are generated through regression analysis using results from the numerical experiments.

The main contributions of our work are (1) to the best of our knowledge, we are the first to identify PO errors other than random yield, and empirically analyze the properties of such errors; (2) using a deterministic inventory model, we have obtained analytical insights about the influence of such PO errors on retail inventory cost and order policies; (3) for the stochastic inventory model with uncertain demand and PO errors, we have conducted a thorough empirical study using real retail data to analyze the impact of PO errors; and, (4) we have proposed procedures as well as

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<sup>3</sup>We have discussed inventory cost parameters at length with our partner retailer. They are unable to provide values for any of the parameters, including the fixed ordering cost.

provided qualitative guidance for a retailer to easily identify SKUs with high potential cost savings or PO error cost with readily available information.

The rest of the paper is organized as follows. Section 2 reviews the related literature and positions our work. Section 3 first introduces our deterministic model to gain insights through an analytical study. We then introduce a more complicated stochastic model which considers uncertainty in demand, PO errors, and rework time. Section 4 discusses the modeling and estimation of PO errors and rework time. Using the estimated parameters, Section 5 performs numerical experiments on the stochastic inventory model and makes insightful observations. It also proposes procedures to easily identify SKUs with high potential cost savings or PO error cost. Section 6 summarizes the main findings of our study and proposes potential directions for future research.

## 2 Literature Review

Among existing papers about random yield, our work is most closely related to the ones that use quality control measures (such as inspection and rework) to deal with the random yield problem. These papers assume that random yield occurs due to production of defective items. The defective items can be detected through inspection and repaired through rework. Under these assumptions, several papers have studied the optimal inventory policy to account for quality control measures: Peters et al. (1988), Porteus (1986), Zhang and Gerchak (1990), Lee (1992), and So and Tang (1995), among others. Our work is most similar to Peters et al. (1988). They employ an inventory model similar to ours except that they assume there is at most one order outstanding at any point in time, while we only assume that orders do not cross in time. Our work differs from Peters et al. (1988) and other papers in several ways. First, these papers are in the production setting, hence, the ways they model random yield are not suitable for PO errors in the retail setting. To the best of our knowledge, our work is the first to identify a set of PO errors in the retail setting that are different from random yield. We have obtained the properties of these errors using data from a large retail DC. Such properties, in particular how PO errors vary with order complexity (i.e., average order quantity per SKU and number of SKUs on a PO), are different from standard assumptions for random yield. Most papers in the random yield literature focus on a single product. They do not need to model the relationship between the random yield and the number of products in a joint order. Secondly, existing papers rely solely on numerical experiments with artificial data to study the influence of rework, while we obtain some analytical properties and carry out numerical experiments with real retail data.

Another closely-related paper in the random yield literature is by Moinzadeh and Lee (1987). They consider the case of an order arriving in two shipments with constant lead time for both shipments. The quantity of the first shipment upon arrival is modeled as a random variable depending on the order quantity. They derive the exact cost function and also provide an approximation based on the assumption that there is at most one order outstanding at any point in time. A solution

algorithm is provided for the approximate cost function under the assumption that the proportion of the first shipment decreases with the order quantity. This assumption is satisfied under the binomial yield and random capacity models, but not for our PO errors. The second shipment in Moinzadeh and Lee (1989) can be thought as being obtained after rework at the vendor. Their rework time, however, does not depend on the error size as for our PO errors.

Our work is an extension of classical inventory management models under continuous time. Hadley and Within (1963) and Zipkin (2000) are excellent references on classical inventory models. In our problem, the PO errors depend on the order quantity, and hence so does the rework time and lead time. There are some papers that discuss how to find the optimal  $(Q, R)$  policy with lot-size dependent lead time, such as those by Kim and Benton (1995) and Hariga (1999). Kim and Benton (1995) also study how the dependence of lead time on lot-size influences the reorder point and order quantity. However, the dependence of lead time in their work is due to the waiting and production process at the manufacturer’s plant, and is modeled as a deterministic linear function of the order quantity. In our study, lead time depends on the order quantity in a more complicated way through rework on errors of a PO at the retail DC, which can not be adequately modeled as a deterministic linear function of the order quantity.

### 3 Inventory Models

Retailers replenish inventory from their vendors using purchase orders (POs). A PO specifies the quantity of each SKU ordered. For example, a PO for a vendor such as Nike might be composed of 100 white jackets and 200 black jackets, as well as 50 white pants and 100 black pants. the fulfillment of POs occasionally arrives to a retail DC with errors. Using data from a large retail chain which operates more than 700 supercenters in a specific geographic region, we observe many types of errors. One type of error is a discrepancy in quantity studied in the well-known random-yield literature. In addition to the discrepancy in quantity, we also observe errors such as “ticket errors.” A ticket error occurs when the vendor of a PO puts an incorrect SKU number or sale price on the tickets attached to some items in a PO. Indeed, any kind of breach of contractual requirements for a PO is a type of PO fulfillment error, which will be referred to as PO errors in brevity henceforth. Another example is the “Did not meet ASN requirements” error. Here, ASN stands for Advanced Shipping Notice. A vendor may forget to send an ASN to the retailer before a PO arrives. Figure 1 shows the PO error percentage breakdown by error type. It is the result of analyzing three month of audit data from our partner retailer. These audit data contains detailed error info for each problematic PO. The labels on the left specify the error type. The percentages to the right show the incidence of each error type among all errors.

Whenever the fulfillment of a PO arrives at a retail DC with errors, the retailer needs to perform rework to address these errors. For example, if an item on a PO has an incorrect ticket attached to it, the retailer needs to replace it with the right one. If the fulfillment of a PO comes in without

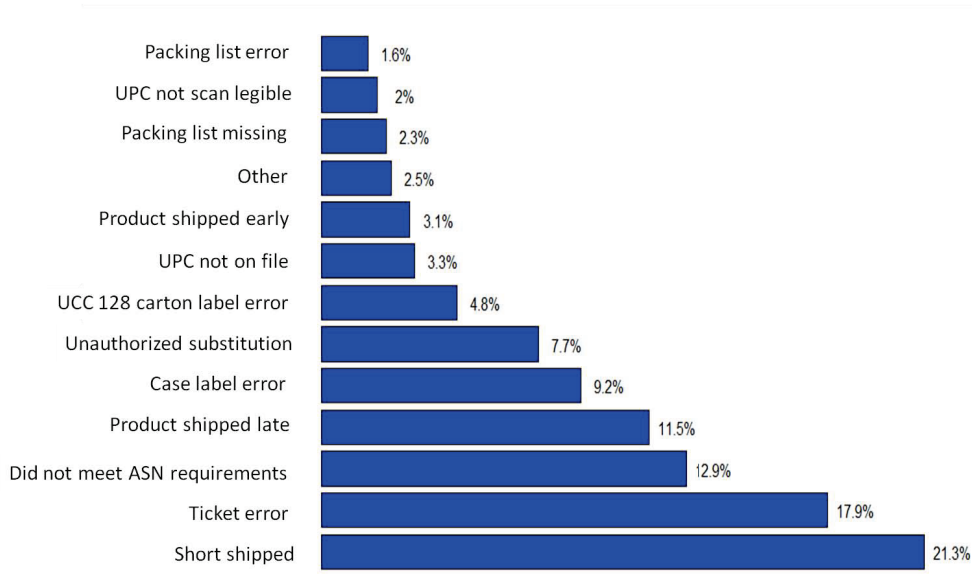


Figure 1: Incidence of PO Error Types

an ASN, the retailer needs to go through each item on the PO in order to determine which PO has arrived. As a result, the retailer incurs additional labor cost, lead time and lead time variability. The additional lead time is composed of two parts, namely the actual rework time and the waiting time at the retail DC. Due to the limited processing capacity at a retail DC, a PO might have to wait in the processing queue for all its previous POs to finish rework. For our partner retailer, the rework time of a PO is on average about 1.6 hours with a maximum of 3.6 days, and the waiting time is typically 2 to 3 days. The standard deviation of the rework time is 5.2 hours. If a PO has to wait at the retail DC for an additional 5.2 hours, it may miss the shipping window from the retail DC to retail stores. In summary, PO errors add cost and complicate retail inventory management. Hence, it is important to take these PO errors into account when building inventory models.

Our inventory models are single SKU models in continuous time. We focus only on correctable PO errors at the retail DC and ignore errors that cannot be corrected. For all the error types at our partner retailer, the bars in solid box in Figure 2 represent errors that are correctable at the retail DC. The bars in dashed box correspond to errors that are not covered under this study, including time errors that cannot be corrected, substitution errors that cannot be handled by a single-item model, and quantity errors that cannot be corrected at the retail DC. Furthermore, quantity errors are already covered in the random yield literature.

### 3.1 Deterministic Inventory Model

Let us assume that demand for a single SKU arrives with constant rate  $\lambda$ . When a retailer places a PO for this single SKU of size  $Q$ , it will receive the order after a constant lead time  $L$ . The

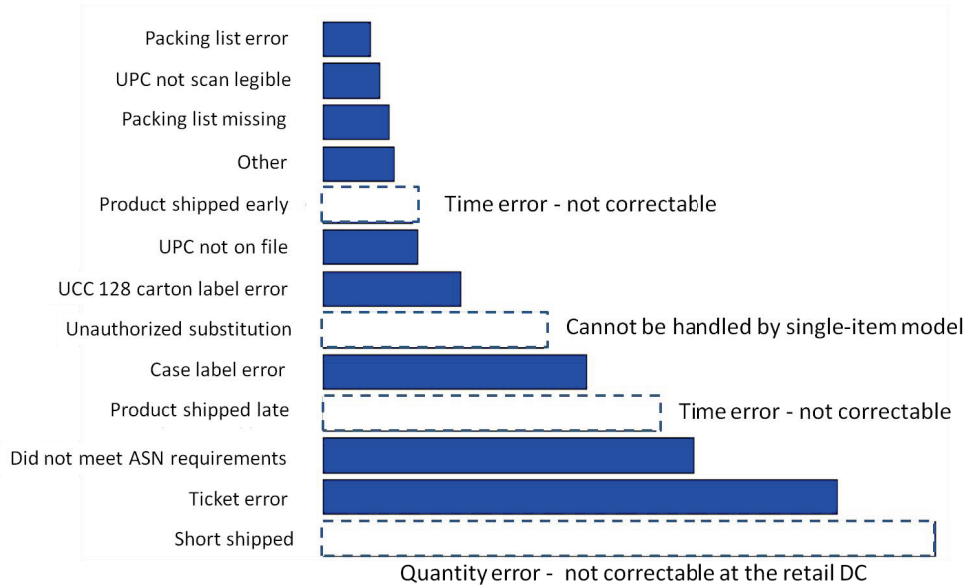


Figure 2: Correctable Errors in Solid Box

order cannot be used until it finishes necessary rework. The rework time of a PO is a function of the error  $W(Q)$ , which is the number of erroneous units and a deterministic function of  $Q$ . Denote the per unit rework time by  $a$ . The rework time per order is  $aW(Q)$ . We make the following assumptions. First, inspection time to check whether a PO has errors is negligible. Second, a PO cannot be used until it finishes rework. In practice, the retailer may use the correct portion of the order during the rework time. Sometimes though, it may defer using it until finishing rework to negotiate with the vendor, settle on a penalty charge, or clarify the source of the error. Third, the retailer does not pay holding cost for the order during the rework time<sup>4</sup>. Since the purchase cost is usually paid after finishing rework, the retailer does not incur holding cost related to the money tied in the inventory during the rework time, which is the major component of holding cost for our partner retailer. Under these three assumptions, the effective lead time is the actual lead time  $L$  plus rework time  $aW(Q)$ . Lastly, we assume that the retailer recovers all the incorrect items of a PO through rework, which is true for many types of PO errors, including ticket errors. Figure 3 illustrates how the inventory level changes over time.

The retailer incurs fixed ordering cost  $K$  per order, holding cost  $h$  per unit per unit time and rework cost  $c_r$  per unit time. Back orders are allowed with backlogging cost  $\pi$  per unit per unit time.<sup>5</sup> Let  $x$  denote the fraction of demand that is backlogged. Then it is optimal for the retailer to place an order  $L + aW(Q)$  before its inventory reaches  $-xQ$  as shown in Figure 3. Define  $\rho(Q)$

<sup>4</sup>This assumption is to simplify the exposition. The model and relevant analysis does not change in substance when the retailer pays holding cost during the rework time

<sup>5</sup>The purpose of the deterministic model is to provide insights for the more realistic stochastic inventory model, which allows back orders.

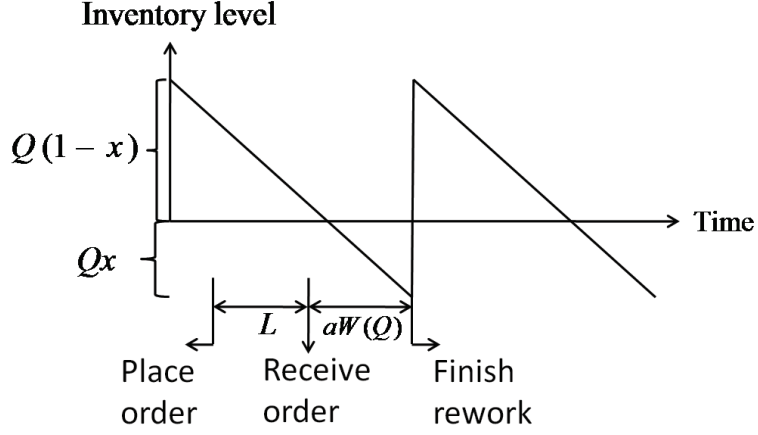


Figure 3: Deterministic Inventory Model

as the error proportion of a PO, which is  $W(Q)/Q$ . The retailer tries to minimize the total cost per unit time  $C_r(Q, x)$ .

$$C_r(Q, x) = \frac{\lambda K}{Q} + c_r a \lambda \rho(Q) + \frac{hQ(1-x)^2}{2} + \frac{\pi Q x^2}{2} \quad (1)$$

Here,  $c_r a \lambda \rho(Q)$  is the rework cost per unit time,  $\frac{hQ(1-x)^2}{2}$  is the holding cost per unit time, and  $\frac{\pi Q x^2}{2}$  is the backlogging penalty cost per unit time.

The total cost per unit time  $C_r(Q, x)$  is convex in the fraction of backlogged demand  $x$ . By setting the first derivative of  $C_r(Q, x)$  with respect to  $x$  to 0, we find that the optimal  $x$  for any given  $Q$  is  $h/(h + \pi)$ . Substituting the optimal  $x$  back into  $C_r(Q, x)$ , we find the system cost as a function of  $Q$  to be:

$$C_r(Q) = \frac{\lambda K}{Q} + c_r a \lambda \rho(Q) + \frac{Q}{2} \frac{h\pi}{h + \pi} \quad (2)$$

Let  $Q_r$  denote the optimal order quantity that minimizes  $C_r(Q)$ , and let  $Q_s$  denote the EOQ order quantity with back orders, which is equal to  $\sqrt{2K\lambda(h + \pi)/h\pi}$  (Axsater (2006)). Theorem 1 explains the influence of PO errors on the optimal order quantity. Proofs for all theorems and propositions in the paper can be found in Appendix A.

**Theorem 1.** *If the error proportion  $\rho(Q)$  decreases in  $Q$ , then the optimal order size in anticipation of PO errors  $Q_r$  is greater than the EOQ order quantity  $Q_s$ .*

We assume the error proportion decreases in  $Q$  because such a relationship is observed empirically.<sup>6</sup> Intuitively, the rework cost per order acts as an additional fixed ordering cost, which increases the optimal order quantity.

<sup>6</sup>Further details of our empirical analysis can be found in Section 4.

We examine the influence of PO errors on the system cost from two perspectives. We introduce two definitions here. The first definition is “PO error cost”, which is equal to the difference between the optimal total cost  $C_r(Q_r)$  and the optimal EOQ cost  $C_s(Q_s)$ , where  $C_s(Q) = \frac{\lambda K}{Q} + \frac{Q}{2} \frac{h}{h + \pi}$ . The PO error cost specifies how much additional cost a retailer incurs due to the existence of PO errors. The second definition is “potential cost savings”, which is equal to the difference between the total cost with EOQ order quantity  $C_r(Q_s)$  and the optimal total cost  $C_r(Q_r)$ . The potential cost savings represents how much a retailer can save by accounting for PO errors. We obtain the following theorems about these two definitions. The assumption about  $\rho(Q)$  in Theorem 3 is based on empirical data analysis of PO errors.

**Theorem 2.** *If the error proportion  $\rho(Q)$  decreases in  $Q$ , then the PO error cost  $C_r(Q_r) - C_s(Q_s)$  decreases in the fixed ordering cost per order  $K$ .*

**Theorem 3.** *If  $\rho(Q) = cQ^{\beta-1}$  for any constant  $c > 0$  and  $0 < \beta < 1$ , then the potential cost savings  $C_r(Q_s) - C_r(Q_r)$  decreases in the fixed ordering cost  $K$ .*

Both the PO error cost and potential cost savings decrease in the fixed ordering quantity  $K$  because as the fixed ordering cost  $K$  increases, the optimal order quantity with rework gets closer to the optimal EOQ order quantity. Thus, the cost difference becomes smaller as  $K$  increases. Theorems 2 and 3 are counter-intuitive in the sense that while the inventory costs  $C_r(Q)$  and  $C_s(Q)$  increase with the fixed ordering cost for any order quantity  $Q$ , the cost impact of PO errors is larger on a retailer with smaller fixed ordering cost. In addition, the cost impact of PO errors is larger for SKUs with smaller fixed ordering cost. Hence, retailers should pay more attention to SKUs with smaller fixed ordering cost when PO errors are present.

**Theorem 4.** *If the error proportion  $\rho(Q)$  decreases in  $Q$ , then the PO error cost  $C_r(Q_r) - C_s(Q_s)$  increases in the backlogging penalty cost  $\pi$  and the unit holding cost  $h$ .*

**Theorem 5.** *If  $\rho(Q) = cQ^{\beta-1}$  for any constant  $c > 0$  and  $0 < \beta < 1$ , then the potential cost savings  $C_r(Q_s) - C_r(Q_r)$  increases in the backlogging penalty cost  $\pi$  and the unit holding cost  $h$ .*

Both the PO error cost and potential cost savings increase in the backlogging penalty cost  $\pi$  or unit holding cost  $h$  because as the backlogging penalty cost  $\pi$  or unit holding cost  $h$  increases, the system inventory costs  $C_r(Q)$  and  $C_s(Q)$  increase for any order quantity  $Q$ . Theorems 4 and 5 tells us the cost impact of PO errors is greater for SKUs with higher purchase cost since both the unit holding cost and the backlogging penalty cost increase in the purchase cost.

**Theorem 6.** *If  $\rho(Q) = cQ^{\beta-1}$  for any constant  $c > 0$  and  $0 < \beta < 1$ , then both the potential cost savings  $C_r(Q_s) - C_r(Q_r)$  and the PO error cost  $C_r(Q_r) - C_s(Q_s)$  increase in the demand rate  $\lambda$ .*

As the demand rate  $\lambda$  increases, the system inventory cost with or without rework ( $C_r(Q)$  or  $C_s(Q)$ , respectively) increases for any order quantity  $Q$ . The differences in cost also increase, including the potential cost savings and PO error cost.

A retailer may need to manage hundreds of thousands of SKUs. It is useful for a retailer to easily identify SKUs with high potential cost savings without adjusting the existing inventory system to account for PO errors. In practice, the retailer may purchase inventory management software from a specialized vendor. It will cost them to adjust and fine-tune the existing inventory system to account for PO errors. Hence, the retailer may only want to do this for SKUs with high potential cost savings, but using only readily available information to select such SKUs. Theorems 3, 5 and 6 provide guidance for retailers to identify SKUs with high potential cost savings. It is also useful for a retailer to easily identify SKUs with high PO error cost without adjusting the existing inventory system to account for PO errors. The retailer may want to collaborate with its vendors to reduce error probability and magnitude. Since the retailer may not want to pay its inventory software vendor to calculate the cost of PO errors for each SKU chosen, it needs a procedure to choose such SKUs based on properties that are easy to obtain. Theorems 2, 4 and 6 provide guidance for retailers to identify SKUs with high PO error cost. More detailed procedures to select SKUs with high potential cost savings or PO error cost will be introduced for the stochastic inventory model in Section 5.

### 3.2 Stochastic Inventory Model

There are three sources of uncertainty in the stochastic setting. First, the demand is uncertain. It is common to assume that the unit demand (i.e., demand per unit time) is normally distributed with mean  $\mu$  and standard deviation  $\sigma$ . The second source is PO error size  $\tilde{W}(Q)$ , which is modeled by a random variable depending on the order quantity  $Q$ . The third source is rework time  $\tilde{t}_r(Q)$ , which is modeled by a random variable depending on the error size and, hence, on the order quantity. Due to its simplicity, it is common in practice to use a  $(Q, R)$  inventory policy, under which an order of size  $Q$  is placed when the inventory position drops to  $R$ . We assume that our retailer employs a  $(Q, R)$  inventory policy. We make several assumptions for the stochastic inventory model in addition to the four assumptions for the deterministic model. First, the excessive demand is backlogged for the stochastic inventory model. Our subject of study is a retail DC. The demand of a retail DC comes from retail stores. In practice, this type of demand is usually backlogged, and the retailer incurs a backlogging penalty cost  $\pi$  per unit per unit time. Second, orders do not cross during the effective lead time, which is equal to the sum of the actual lead time  $L$  and rework time  $\tilde{t}_r(Q)$ . Third, the demand during the effective lead time is normally distributed with mean  $\mu(Q) = \mu E(L + \tilde{t}_r(Q))$  and standard deviation  $\sigma(Q) = (\sigma^2 E(L + \tilde{t}_r(Q)) + \mu^2 Var(L + \tilde{t}_r(Q)))^{\frac{1}{2}}$ . The models for the PO error size  $\tilde{W}(Q)$  and rework time  $\tilde{t}_r(Q)$  will be obtained after empirical data analysis in Section 4. The expressions for  $\mu(Q)$  and  $\sigma(Q)$  will be updated in more details then. Figure 4 shows how the inventory level evolves over time for the stochastic inventory model.

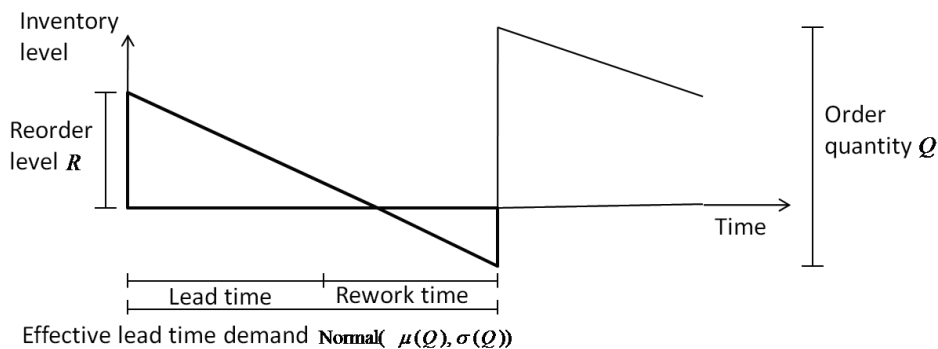


Figure 4: Stochastic Inventory Model

The retailer wants to minimize the total cost per unit time  $C_r(Q, R)$  as following:

$$(K + c_r E(\tilde{t}_r(Q))) \frac{\mu}{Q} + h(R + \frac{Q}{2} - \mu(Q)) + (h + \pi) \frac{\sigma^2(Q)}{Q} (H(\frac{R - \mu(Q)}{\sigma(Q)}) - H(\frac{R + Q - \mu(Q)}{\sigma(Q)}))$$

Here  $H(x) = \frac{1}{2}((x^2 + 1)(1 - \Phi(x)) - x\varphi(x))$ <sup>7</sup>, where  $\Phi(x)$  and  $\varphi(x)$  are respectively the standard normal cumulative distribution and density function. The stochastic model is more difficult to analyze analytically than the deterministic model. Hence, we mainly use numerical experiments to analyze it. The numerical experiments will be introduced in Section 5. In these experiments, we find the optimal  $(Q, R)$  by solving a system of nonlinear first order equations using Newton's method. The starting value for  $Q$  is the optimal order quantity in the deterministic setting. The starting value for  $R$  is the expected effective lead time demand. The ordering policy  $(Q, R)$  found in this way is only a local optimal solution without proving the unimodality of the total cost per unit time  $C_r(Q, R)$  in  $(Q, R)$ . In our numerical experiments, we have observed the unimodality of  $C_r(Q, R)$ . We assume here that the total cost per unit time  $C_r(Q, R)$  is unimodal so that the locally optimal ordering policy is also globally optimal.

## 4 Estimation of PO Errors and Rework Time

In order to define appropriate parameters for our numerical experiments, we use actual data from a retail chain to model PO errors. Specifically, we evaluate whether, and to what extent, PO errors depend on the order quantity. We also assess whether, and how, rework time depends on PO errors.

<sup>7</sup>Here we define  $H(x)$  in the same way as in Axsater (2006).

## 4.1 Data Description

Our data was drawn from Omega (a well-known discount retailer) that operates more than 700 stores. The products Omega carries include apparel, electronics, and housewares, which compose around 60% of its sales. The other 40% are from food items. We focus only on the non-food items. The data we collected contains observations of each PO, including the quantity purchased for each item within the PO, during the course of an entire year. The data also records the order date and arrival date of a PO, which can be used to derive lead time for the PO. We also capture three months of audit data, which record the errors and rework time for each PO.

## 4.2 Regression Models and Estimation for PO Errors

At our partner retailer Omega, errors are tracked at the PO level. We define the error size of a PO as the sum of errors for all SKUs on the PO. There are many different types of potential PO errors that a PO might have, as shown in Figure 1. We do not differentiate between different error types. A PO error is a generic error, which can be of any error type. In addition, there might be more than one type of error observed in a particular PO. For our partner retailer Omega, around 45% of POs with errors have more than one type of PO error. The error size of such a PO is defined to be the sum of errors of all types.

The more complex a PO, the more likely it has an error and the larger the expected error size. We use two measures, namely the average order quantity per SKU  $Q$  and the number of SKUs on a PO  $n$ , to characterize the complexity of a PO. For a PO composed of 100 white jackets and 50 white pants, the average order quantity of this PO is 75, and the number of SKUs is 2. PO error size depends on these two measures.

Each PO belongs to a specific vendor and department. A department contains many SKUs. Examples of departments include “Children” and “Shoes”. A vendor can sell to more than one department. It is expected that PO errors vary from vendor to vendor and department to department, and that PO errors within the same vendor or department will tend to be similar to each other. We use cross-classified hierarchical linear modeling (HLM) that allows observations to be nested with two higher-level categories, namely vendors and departments, to account for the multilevel structure of our data (Raudenbush and Bryk (2001)). To estimate these HLM models, we use the “lmer” function in R version 2.8.1 and employ a maximum likelihood estimation technique.<sup>8</sup>

Since the percentage of POs with errors is very low (5.41%), we model PO error size  $\tilde{W}_{ilj}$  as follows:

$$E(\tilde{W}_{ilj}|Q_{lij}, n_{lij}) = P(\tilde{W}_{ilj} > 0|Q_{lij}, n_{lij}) \times E(\tilde{W}_{ilj}|\tilde{W}_{ilj} > 0, Q_{lij}, n_{lij}) \quad (3)$$

Here  $E(\tilde{W}_{ilj}|Q_{lij}, n_{lij})$  is the expected error size of PO  $l$  ( $l = 1, \dots, m_{ij}$ ) from vendor  $i$  ( $i = 1, \dots, 112$ ) and department  $j$  ( $j = 1, \dots, 14$ ) with average order quantity  $Q_{lij}$  and number of SKUs  $n_{lij}$ . The

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<sup>8</sup>Consistent results are obtained using Proc Mixed in SAS v.9.1.

expression  $P(\tilde{W}_{lij} > 0 | Q_{lij}, n_{lij})$  represents the probability of an error for PO  $l$  from vendor  $i$  and department  $j$  with average order quantity  $Q_{lij}$  and number of SKUs  $n_{lij}$ . The expression  $E(\tilde{W}_{lij} | \tilde{W}_{lij} > 0, Q_{lij}, n_{lij})$  is the expected error size of PO  $l$  from vendor  $i$  and department  $j$ , given that there is an error, with average order quantity  $Q_{lij}$  and number of SKUs  $n_{lij}$ .

When using HLM models, we first need to test the existence of random effects using a null model. We then include our predictors and estimate the conditional model. A logistic model is used for the probability of an error. The null model for the probability of an error  $P(\tilde{W}_{lij} > 0)$  is the following:

$$\text{logit}(P(\tilde{W}_{lij} > 0)) = b_0 + c_{00i}^p + d_{00j}^p + e_{lij}^p \quad (4)$$

where  $b_0$  is a fixed intercept parameter. The random effect of vendor  $i$  is  $c_{00i}^p \sim N(0, \tau_{c00}^p)$ . The random effect of department  $j$  is  $d_{00j}^p \sim N(0, \tau_{d00}^p)$ . The random PO effect is  $e_{lij}^p \sim N(0, \sigma_p^2)$ . We use a Chi-square test to test the significance of the random effects of vendor and department, and find that they are both significant at the confidence level of 95%.

The conditional model for the probability of an error  $P(\tilde{W}_{lij} > 0)$  with predictors  $Q_{lij}$  and  $n_{lij}$  is shown in equation (5). We have included the predictors in their natural log forms to induce linearity. We find that this is the most effective way after analyzing the raw data. We employ the model

$$\text{logit}(P(\tilde{W}_{lij} > 0 | Q_{lij}, n_{lij})) = b_0 + c_{00i}^p + d_{00j}^p + e_{lij}^p + b_1 \times \log(Q_{lij}) + b_2 \times \log(n_{lij}) \quad (5)$$

in which  $b_0, b_1, b_2$  are fixed PO-level coefficients. We assume that  $b_1$  and  $b_2$  are fixed across vendors and departments, rather than randomly varying, because the number of POs under some vendor-department combinations is small.

We filter out all POs under the vendor-department combinations with more than 2 error records in order to estimate  $P(\tilde{W}_{lij} > 0)$ . After filtering, we have 3788 POs. Table 1 shows the summary statistics across all of these POs for the regression variables. In the table, ‘‘StDev’’ stands for standard deviation.

Table 1: Summary Statistics for Error Likelihood Model

Variable	Min	Max	Median	Mean	StDev	Yes	No
$\tilde{W}_{lij} > 0$						536	3,252
$Q_{lij}$	1	125,100	810	2,253	5,650.29		
$n_{lij}$	1	592	12	27.02	46.44		

The null model for the error size  $\tilde{W}_{lij}$  given that there is an error is as follows:

$$\tilde{W}_{lij} | \tilde{W}_{lij} > 0 = \alpha + c_{00i}^s + d_{00j}^s + e_{lij}^s \quad (6)$$

where  $\alpha$  is a fixed intercept parameter. The random effect of vendor  $i$  is  $c_{00i}^s \sim N(0, \tau_{c00}^s)$ . The random effect of department  $j$  is  $d_{00j}^s \sim N(0, \tau_{d00}^s)$ . The random PO effect is  $e_{lij}^s \sim N(0, \sigma_s^2)$ . We

use a Chi-square test to test the significance of the random effects of vendor and department. We find that the random effect of vendor is significant with  $p$  value smaller than 0.001, and the random effect of department is not statistically significant. Hence, we do not include the random effect of department in the following conditional model. Instead, we account for heterogeneity of POs under different departments by using fixed effects (e.g., dummy variables to represent each department).

We use a log-log model to determine the error size given that there is an error. The conditional model for the error size  $\tilde{W}_{lij}$  given that there is an error with predictors  $Q_{lij}$  and  $n_{lij}$  is as follows:

$$\log(\tilde{W}_{lij} | \tilde{W}_{lij} > 0, Q_{lij}, n_{lij}) = \alpha_j + c_{00i}^s + d_{00j}^s + e_{lij}^s + \beta \times \log(Q_{lij}) + \gamma \times \log(n_{lij}) \quad (7)$$

Here  $\alpha_j, \beta, \gamma$  are fixed PO-level coefficients. The constant terms  $\alpha_j$  are again used to account for the heterogeneity of POs under different departments. We assume that  $\beta$  and  $\gamma$  are fixed across vendors and departments, rather than randomly varying, because the number of POs under some vendor-department combinations is small. Table 2 shows the summary statistics across POs with positive error size for the dependent and independent variables of the error size model. In total, we have 536 POs with a nonzero error.

Table 2: Summary Statistics for Conditional Error Size Model

Variable	Min	Max	Median	Mean	StDev
$\tilde{W}_{lij}$	1	34,440	39	317	1,760.25
$Q_{lij}$	11	80,620	1,440	2,253	6,439.31
$n_{lij}$	1	592	12	27.02	63.53

Table 3 shows the estimation results for the likelihood of an error. The deviance (i.e. negative twice the log-likelihood) of the fit is 2899<sup>9</sup>. From the regression results, we can see that the likelihood of an error increases in both the average order quantity and in the number of SKUs on a PO. This confirms the intuition that the more complex a PO is, the more likely there is an error. The estimates for  $\tau_{c00}^p$ ,  $\tau_{d00}^p$ , and  $\sigma_p^2$  are respectively 0.450, 0.071, and 1.

Table 3: Error Likelihood Model Estimation Results ( $n = 3788$ )

Regression Coefficients	Estimate	Standard Error	$z$ value	$p$ value
$b_0$	-3.344	0.306	-10.916	< 0.001
$b_1$ ( $\log(Q_{lij})$ )	0.179	0.040	4.531	< 0.001
$b_2$ ( $\log(n_{lij})$ )	0.369	0.055	6.659	< 0.001

Table 4 shows the estimation results for the error size conditioning on the existence of an error. The  $p$  values in the table are calculated using the “pvals.fnc” function in the “languageR” package. Note that here we only show the regression coefficients for two of the department dummy variables.

<sup>9</sup>The deviance is a measure of goodness of fit for the hierarchical linear models estimated through maximum likelihood estimation technique. The larger the deviance, the worse the fit of the model.

The deviance of the fit is 2231. From the regression results, we can see that the magnitude of an error increases in both the average order quantity and in the number of SKUs on a PO. This confirms the intuition that the more complex a PO is, the larger the expected error size is. We can also see that each department has a different estimated constant term  $\alpha_j$ . The estimates for  $\tau_{c00}^s$  and  $\sigma_s^2$  are respectively 0.354 and 3.485.

Table 4: Conditional Error Size Model Estimation Results ( $n = 536$ )

Regression Coefficients	Estimate	Standard Error	$t$ value	$p$ value
$\beta (\log(Q_{lij}))$	0.433	0.064	6.754	< 0.001
$\gamma (\log(n_{lij}))$	0.239	0.092	2.589	0.010
$\alpha_1$	1.278	0.594	2.151	0.032
$\alpha_2$	1.159	0.492	2.353	0.020

### 4.3 Regression Model and Estimation for Rework Time

The extra time a PO with errors stays at a retail DC is composed of the waiting time and the actual rework time. A PO with errors may have to wait at the retail DC for its previous POs to finish rework. Although we do not have data to determine the waiting time for each PO, we are informed by our partner retailer Omega that the waiting time for a PO is typically 2 to 3 days. In addition, we have the rework time for each PO. The rework time for POs with zero errors is zero. Since the percentage of POs with errors is very low (around 5%), we only model rework time for POs with errors. Such rework time should depend on the error size of a PO.

We first test the existence of random effects using a null model. The null model for rework time  $\tilde{t}_{lij}$  conditioning on that there is an error is as follows:

$$\tilde{t}_{lij} | \tilde{W}_{lij} > 0 = \phi_0 + c_{00i}^r + d_{00j}^r + e_{lij}^r \quad (8)$$

where  $\phi_0$  is a fixed intercept parameter, and  $\tilde{W}_{lij}$  denotes the error size of a PO. The random effect of vendor  $i$  is  $c_{00i}^r \sim N(0, \tau_{c00}^r)$ . The random effect of department  $j$  is  $d_{00j}^r \sim N(0, \tau_{d00}^r)$ . The random PO effect is  $e_{lij}^r \sim N(0, \sigma_r^2)$ . We use a Chi-square test to test the significance of the random effects of vendor and department. We find that the random effect of vendor is significant with  $p$  value smaller than 0.001, and the random effect of department is not statistically significant. Hence we do not include the random effect of departments in the following conditional model. Instead, we account for heterogeneity of POs under different departments using using fixed effects (e.g., dummy variables to represent each department).

We use a linear model for rework time conditioning on that there is an error, with per unit rework time  $a_j + c_{00i}^r$  varying by vendor  $i$  and department  $j$ . Intuitively, the per unit rework time should also vary by the error type. However, we have already accounted for the heterogeneity of the per unit rework time for different vendor-department combinations, and certain vendor-department

combinations are usually associated with certain types of errors. It is therefore redundant to consider the variability of the per unit rework time by error types in addition to the variability between different vendor-department combinations. The regression model for the rework time is as follows:

$$\tilde{t}_{lij}|\tilde{W}_{lij} > 0 = a_j \times \tilde{W}_{lij} + c_{00i}^r \times \tilde{W}_{lij} + e_{lij}^r \quad (9)$$

Note that we do not include an intercept term for the rework time model. The intercept term can be interpreted as the set up time for rework. The rework needed for POs with errors does not require set up time. For example, the rework needed for the error type such as “Did not meet ASN requirements” is to go through each item on a PO to figure out which PO it is, a process which does not require a set up. In addition, when testing multiple models with different functional forms (including a model with an intercept term), the no intercept model generates the best fit. Hence, we find that it is more appropriate not to include an intercept term. We have 465 POs with errors for the estimation of rework time model. The following table shows the summary statistics for the regression variables. The unit for the rework time is measured in minutes.

Table 5: Summary Statistics for Rework Time Model

Variable	Min	Max	Median	Mean	StDev
$\tilde{t}_{lij}$	0	5235	30	96.03	314.32
$\tilde{W}_{lij}$	1	34440	35	305.2	1,822.97

Table 6 shows the estimation results for rework time. We show  $a_j$  only for two departments for the purpose of illustration. The deviance of the fit is 6155. The estimates for  $\tau_{c00}^r$  and  $\sigma_r^2$  are respectively 1.762 and 18,969.465.

Table 6: Rework Time Model Estimation Results ( $n = 465$ )

Regression Coefficients	Estimate (min/unit)	Standard Error	$t$ value	$p$ value
$a_1$	0.738	0.251	2.940	0.003
$a_2$	0.316	0.325	0.973	0.331

#### 4.4 Error Size and Rework Time for a PO of a single SKU

We have obtained forecast models for PO error size with average order quantity  $Q$  and number of SKUs  $n$  as predictors. Now, we will derive expressions for the error mean  $E(\tilde{W}(Q))$  for a PO of a single SKU using these forecast models.

We use the error for a PO of a single SKU to estimate the error for that SKU. According to equation (3), for a PO of a single SKU with order quantity  $Q$ , the PO error mean  $E(\tilde{W}(Q))$  is  $P(\tilde{W}(Q) > 0|Q, n = 1) \times E(\tilde{W}(Q)|\tilde{W}(Q) > 0, Q, n = 1)$ . According to equation (5), the probability of an error for a PO of a single SKU  $P(\tilde{W}(Q) > 0|Q, n = 1)$  is  $\frac{e^{(b_0+b_1 \log(Q))}}{1 + e^{(b_0+b_1 \log(Q))}}$ , which is henceforth

denoted by  $p(Q)$ . Note that the number of SKUs for a PO of a single SKU is 1. According to equation (7), given that there is an error, the error size for a PO of a single SKU with order quantity  $Q$  has a log-normal distribution with mean equal to  $e^{(\alpha+\sigma_s^2/2)}Q^\beta$ . Hence the PO error mean  $E(\tilde{W}(Q))$  is  $p(Q)e^{(\alpha+\sigma_s^2/2)}Q^\beta$ . The derivation of the PO error mean is given in Appendix B.

The error proportion  $\rho(Q)$  for the deterministic model is considered to be the expected error proportion  $E(\tilde{W}(Q))/Q$ . The following proposition states a sufficient condition on the regression coefficient estimates of the PO error forecast models for  $E(\tilde{W}(Q))/Q$  to decrease in the order quantity  $Q$ .

**Proposition 7.** *If  $\beta + b_1 \leq 1$ , then  $E(\tilde{W}(Q))/Q$  decreases in the order quantity  $Q$ .*

From the regression results of the probability of an error and conditional error size, we can see that the estimate for  $b_1$  is 0.179, and that the estimate for  $\beta$  is 0.433. The sufficient condition in Proposition 7 is satisfied. Hence, the resulting error proportion  $\rho(Q)$  decreases in the order quantity  $Q$ , as assumed in Theorems 1 and 2. If we assume that the probability of an error  $p(Q)$  is equal to a constant  $p$ , which does not depend on the order quantity  $Q$ , then  $\rho(Q)$  is equal to  $pe^{(\alpha+\sigma_s^2/2)}Q^{\beta-1}$ . Define  $c$  as  $pe^{(\alpha+\sigma_s^2/2)}$ . We can see that  $\rho(Q) = cQ^{\beta-1}$  for some constant  $c > 0$  and  $0 < \beta < 1$ , as assumed in Theorems 3, 4, 5 and 6.

We have also gotten the forecast model for the rework time conditioning on there is an error. Now we can derive the rework time mean  $E(\tilde{t}_r(Q))$  and variance  $Var(\tilde{t}_r(Q))$  for a PO of a single SKU with order quantity  $Q$ . The rework time mean  $E(\tilde{t}_r(Q))$  is  $ap(Q)e^{(\alpha+\sigma_s^2/2)}Q^\beta$ , and the rework time variance  $Var(\tilde{t}_r(Q))$  is  $a^2e^{2\alpha+\sigma_s^2}Q^{2\beta}(e^{\sigma_s^2} - p(Q))p(Q) + \sigma_r^2p(Q)$ . The derivation of the rework time mean and variance is given in Appendix B.

Now we can derive expressions for the effective lead time demand mean  $\mu(Q)$  and the standard deviation  $\sigma(Q)$ . For a PO of a single SKU with order quantity  $Q$ , the effective lead time demand mean  $\mu(Q)$  is given by the following expression.

$$\begin{aligned}\mu(Q) &= \mu E(L + \tilde{t}_r(Q)) \\ &= \mu L + \mu E(\tilde{t}_r(Q)) \\ &= \mu L + \mu ap(Q)e^{(\alpha+\sigma_s^2/2)}Q^\beta\end{aligned}$$

The effective lead time demand standard deviation  $\sigma(Q)$  is given by the following expression. Note that POs from each vendor and department combination has different estimates for the error likelihood parameter  $\alpha$ , the conditional error magnitude parameter  $b_0$  and the unit rework time parameter  $a$ .

$$\begin{aligned}\sigma(Q) &= (\sigma^2 E(L + \tilde{t}_r(Q)) + \mu^2 Var(L + \tilde{t}_r(Q)))^{\frac{1}{2}} \\ &= (\sigma^2 L + \sigma^2 E(\tilde{t}_r(Q)) + \mu^2 Var(\tilde{t}_r(Q)))^{\frac{1}{2}} \\ &= (\sigma^2 L + \sigma^2 ap(Q)e^{(\alpha+\sigma_s^2/2)}Q^\beta + \mu^2 a^2 p(Q)e^{(2\alpha+\sigma_s^2)}(e^{\sigma_s^2} - p(Q))Q^{2\beta} + \mu^2 \sigma_r^2 p(Q))^{\frac{1}{2}}\end{aligned}$$

## 5 Numerical Experiments

We use numerical experiments to study the influence of PO errors on a retailer's inventory policy and system cost for the stochastic inventory model. In order to perform the numerical experiments, in addition to the data about PO errors and rework time used for parameter estimation in Section 4, we also need SKU demand data and inventory cost parameters. Sales or inventory data is not available to estimate the demand for each SKU. Therefore, we approximate demand in a particular month using the sum of order arrivals during the month. If no orders arrive during a month for a SKU, that month is not included to calculate monthly demand mean and standard deviation. In this case, this SKU is a seasonal product. If we assume demand for such months is zero, we will overestimate the standard deviation of the demand. We spoke at length with our partner retailer about the inventory cost parameters. They are unable to provide estimates for these values, especially for the fixed ordering cost  $K$ . They provide us with the average purchase cost breakdown by department, from which we can estimate the holding cost  $h$ . The selected SKUs in the numerical experiments are from fourteen different departments. To the best of our knowledge, the literature provides no standard method to choose inventory cost parameters for numerical experiments. We have run experiments with the following cost parameter settings: fixed ordering cost ranges from \$10 to \$100 per order with an increment of \$10; holding cost per year ranges from 5% to 30% of the purchase cost with an increment of 5%; backlogging penalty cost per year ranges from 5 to 40 times the holding cost per year with an increment of 5.

In the numerical experiments, we build one standard  $(Q, R)$  inventory model and one inventory model with rework for each SKU with enough data for estimation. There are around 13,500 such SKUs in our retail data set<sup>10</sup>. We assume that there is no joint fixed ordering cost when ordering several SKUs together. Federgruen and Zheng (1992) and Rosling (1999) discuss efficient solution algorithms to find the optimal  $(Q, R)$  for the standard stochastic inventory model with a  $(Q, R)$  policy. We use the iterative algorithm in Rosling (1999) to find optimal  $(Q_s, R_s)$  for the standard inventory model. We provide a brief description of the algorithm. Start with  $Q$  as the EOQ order quantity. Find the optimal reorder point  $R$  for a given  $Q$  by solving the equation generated from setting  $\frac{\partial C_s(Q, R)}{\partial R}$  equal to zero, where  $C_s(Q, R)$  is the inventory cost function for the standard inventory model. Get the order quantity  $Q$  for the next iteration using an updating formula generated from setting  $\frac{\partial C_s(Q, R)}{\partial Q}$  equal to zero. Repeat this process until the inventory cost  $C_s(Q, R)$  converges. The solution algorithm used to find the optimal  $(Q_r, R_r)$  for the stochastic inventory model that accounts for PO errors is as described in Section 3.2. Note that without proving the

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<sup>10</sup>There are around 93,000 SKUs in the one year data set that records the order quantity for each item within each PO, as well as the order date and receive date for each PO. There are around 14,000 SKUs in the three months of audit data which records the error and rework time for each PO.

unimodality of the system inventory cost with rework  $C_r(Q, R)$ , the optimal solution  $(Q_r, R_r)$  found using the previous algorithm is only locally optimal. Therefore, in the following numerical experiments, the potential cost savings calculated as  $C_r(Q_s, R_s) - C_r(Q_r, R_r)$  is only a lower bound of the achievable potential cost savings. The PO error cost calculated as  $C_r(Q_r, R_r) - C_s(Q_s, R_s)$  is an upper bound of the true PO error cost. However, we have observed the unimodality of the cost function  $C_r(Q, R)$  in our numerical experiments for various parameter settings. Therefore, these bounds should reflect the actual potential cost savings and the true PO error cost.

## 5.1 Influence of PO Errors

We have run 480 experiments for all the selected SKUs with inventory cost parameters as described above. Table 7 shows the summary statistics of the potential cost savings for all the selected SKUs for some representative experiments. In the table, the column labeled “h percent” is the annual holding cost as a percentage of the purchase cost. The columns “Min”, “Median”, “Mean” and “Max” are the summary statistics for monthly potential cost savings at the SKU level. The column “Yearly Total” is the yearly total potential cost savings of all the selected SKUs. For SKUs with zero sales for some months, the yearly total potential cost savings is calculated by multiplying the monthly savings with the number of months with nonzero sales. The column “Yearly per SKU” is the average yearly potential cost savings per SKU. From this table, we can see that the retailer can achieve potential cost savings from \$0.65 to \$19.63 per year per SKU by accounting for PO errors that lead to rework. It is about 9% to 260% of the average purchase cost per SKU. The potential cost savings varies with inventory cost parameters. More specifically, the average yearly potential cost savings per SKU decreases in the fixed ordering cost, and increases in the holding cost percentage as well as in the backlogging penalty cost. In addition, the average yearly potential cost savings per SKU is most sensitive to the holding cost percentage, and least sensitive to the fixed ordering cost<sup>11</sup>.

Table 8 shows the summary statistics of PO error cost in dollar amounts among all the selected SKUs for some representative experiments. The columns in the table have similar meanings as those in Table 7 with potential cost savings replaced by PO error cost. From this table, we can see that PO errors cost the retailer from \$11.19 to \$51.05 per year per SKU. This is about 150% to 670% of the average purchase cost per SKU. The PO error cost varies with inventory cost parameters. More specifically, the yearly PO error cost per SKU decreases in the fixed ordering cost, and increases in the holding cost percentage as well as in the backlogging penalty cost. In addition, the yearly PO error cost is most sensitive to the holding cost percentage, and least sensitive to the fixed ordering cost. Currently, the retailer charges its vendors about \$25 per SKU for these PO errors. It might undercharge or overcharge its vendors depending on its inventory cost parameters. When

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<sup>11</sup>It is the most sensitive in the sense that  $\frac{\Delta \text{potential cost savings}}{\text{potential cost savings}} / \frac{\Delta \text{cost parameter}}{\text{cost parameter}}$  is the largest for the holding cost percentage.

Table 7: Potential Cost Savings Summary Statistics over Selected SKUs (in Dollars)

h percent	K	$\pi/h$	Min	Median	Mean	Max	Yearly Total	Yearly per SKU
5%	10	5	0.00	0.04	0.28	133.37	25,147	1.87
5%	100	5	0.00	0.01	0.10	43.38	8,739	0.65
5%	10	40	0.00	0.12	0.54	140.09	50,659	3.77
5%	100	40	0.00	0.06	0.28	83.69	27,310	2.03
15%	10	40	0.00	0.41	1.40	260.64	132,704	9.86
15%	100	40	0.00	0.24	0.94	256.54	90,964	6.76
20%	10	5	0.00	0.18	0.69	134.72	62,460	4.64
20%	100	5	0.00	0.08	0.38	93.67	35,184	2.62
30%	10	5	0.00	0.29	0.99	142.48	89,385	6.64
30%	100	5	0.00	0.14	0.61	141.04	55,768	4.15
30%	10	40	0.00	0.88	2.79	567.37	264,063	19.63
30%	100	40	0.00	0.57	2.05	516.87	198,120	14.73

the holding cost is 15% of the purchase cost, the fixed ordering cost is \$10 per order, and the backlogging penalty cost is 25 times the holding cost, the retailer undercharges its vendors for these PO errors. When the holding cost is 5% of the purchase cost, the fixed ordering cost is \$10 per order, and the backlogging penalty cost is 40 times the holding cost, the retailer overcharges its vendors for these PO errors.

Table 8: PO Error Cost Summary Statistics over Selected SKUs (in Dollars)

h percent	K	$\pi/h$	Min	Median	Mean	Max	Yearly Total	Yearly per SKU
5%	10	5	0.01	0.75	2.34	305.49	207,806	15.45
5%	100	5	0.01	0.48	1.69	273.28	150,555	11.19
5%	10	40	0.01	0.90	2.71	329.98	243,489	18.10
5%	100	40	0.01	0.62	2.03	293.74	183,651	13.65
15%	10	25	0.01	1.54	4.46	619.67	412,335	30.65
15%	100	25	0.01	1.14	3.55	612.82	331,397	24.63
20%	10	5	0.01	1.47	4.28	571.35	394,710	29.34
20%	100	5	0.01	1.01	3.33	564.59	309,266	22.99
30%	10	20	0.02	2.40	6.68	1,182.34	630,986	46.90
30%	100	20	0.01	1.81	5.57	1,174.32	530,872	39.46
30%	10	40	0.03	2.62	7.25	1,331.19	686,880	51.05
30%	100	40	0.01	2.03	6.13	1,322.95	586,419	43.59

The optimal ordering policy is influenced by PO errors. In the deterministic model, we can show analytically that the optimal order size  $Q_r$  that accounts for PO errors is greater than the optimal EOQ order quantity  $Q_s$ . In the stochastic case, we want to know if a similar property holds, that is, if the optimal order size that accounts for PO errors is greater than the optimal order quantity for the standard  $(Q, R)$  model. For all the selected SKUs in the previous 480 experiments,  $Q_r$  is

always greater than  $Q_s$ . Rework adds additional lead time and lead time variability. As a result, we would expect the reorder point with rework to be higher than the standard one. For 99.36% of test cases,  $R_r$  is no less than  $R_s$ . There are cases to the contrary.

As observed from Tables 7 and 8, both average yearly potential cost savings and PO error cost per SKU decrease in the fixed ordering cost, and increase in the holding cost percentage as well as in the backlogging penalty cost. In the deterministic model, we can also show analytically that both potential cost savings and PO error cost decrease in the fixed ordering cost  $K$ , increase in the holding cost  $h$ , and increase in the backlogging penalty cost  $\pi$  for every individual SKU, not just the average value over all SKUs. Intuitively, the same property should still hold in the stochastic case. We have constructed several experiments to test our intuition.

To test whether potential cost savings and PO error cost decrease in the fixed ordering cost, we set the holding cost percentage to be 30%, and the backlogging penalty cost to be 20 times the holding cost. We vary the fixed ordering cost from \$10 to \$100 per order with an increment of \$10. We compare the potential cost savings of each SKU with the fixed ordering cost of  $\$K$  and  $\$(K + 10)$  per order for  $K$  from \$10 to \$90. We find that potential cost savings with lower fixed ordering cost per order  $K$  is greater than that with a higher  $K$  in 96.15% of test cases. We also compare the PO error cost of each SKU with the fixed ordering cost of  $\$K$  and  $\$(K + 10)$  per order for  $K$  from \$10 to \$90. We find that the PO error cost with lower fixed ordering cost per order  $K$  is greater than that with a higher  $K$  in 98.93% of test cases. To test whether potential cost savings and PO error cost increase in the holding cost percentage, we set the fixed ordering cost to be \$100 per order, and the backlogging penalty cost to be 20 times the holding cost. We vary the holding cost percentage from 5% to 30% of the purchase cost with an increment of 5%. We compare the potential cost savings of each SKU with the holding cost percentage of  $h$  and  $(h + 5\%)$  for  $h$  from 5% to 25%. We find that potential cost savings with higher holding cost percentage is greater than that with a lower one in 98.17% of test cases. We also compare the PO error cost of each SKU with the holding cost percentage of  $h$  and  $(h + 5\%)$  for  $h$  from 5% to 25%. We find that PO error cost with higher holding cost percentage is greater than that with a lower one in 99.56% of test cases. To test whether potential cost savings and PO error cost increase in the backlogging penalty cost, we set the holding cost percentage to be 30%, and the fixed ordering cost to be \$100 per order. We vary the backlogging penalty from 5 to 40 times the holding cost with an increment of 5 times. We compare the potential cost savings of each SKU with the backlogging penalty cost of  $\pi$  and  $\pi + 5$  times the holding cost for  $\pi$  from 5 to 35. We find that potential cost savings with higher backlogging penalty cost is greater than that with a lower one in 98.82% of test cases. We also compare the PO error cost of each SKU with the backlogging penalty cost of  $\pi$  and  $\pi + 5$  times the holding cost for  $\pi$  from 5 to 35. We find that PO error cost with higher backlogging penalty cost is greater than that with a lower one in 99.47% of test cases.

The qualitative properties above regarding how potential cost savings or PO error cost varies with inventory cost parameters can guide a retailer in identifying SKUs with higher potential cost

savings in order to adjust inventory policies or higher PO error cost in order to reduce error size. However, a retailer may need to manage hundreds of thousands of SKUs, and hence needs a more readily usable procedure to pick desirable SKUs from this tremendous quantity of SKUs. The following subsection proposes predictive models to estimate potential cost savings or PO error cost based on readily available properties of SKUs through regression analysis.

## 5.2 Identify SKUs with High Potential Cost Savings or PO Error Cost

Both the standard inventory cost  $C_s(Q, R)$  and the inventory cost with rework  $C_r(Q, R)$  vary in the inventory cost parameters ( $K$ ,  $h$ , and  $\pi$ ), the SKU unit time demand parameters (unit time demand mean  $\mu$  and standard deviation  $\sigma$ ), and the SKU lead time  $L$ . In addition, the inventory cost with rework  $C_r(Q, R)$  also varies in the PO error related parameters ( $b_0$  and  $\alpha$ )<sup>12</sup>, and the SKU unit rework time  $a$ . Hence, we would expect both the potential cost savings and PO error cost for any SKU to be functions of these parameters. Analysis of raw data suggests a nonlinear relationship between the outcome variable (potential cost savings or PO error cost), and each of these parameters. We transform the outcome variable and some of these parameters (including  $K$ ,  $h$ ,  $\pi$ ,  $\mu$ , and  $L$ ) into their natural log form. We find the natural log transformation to be most effective in inducing linearity. Because the potential cost savings or PO error cost may be zero for some SKUs, we add 0.01 to it for the existence of its natural log form. We pick the number 0.01 because it is the smallest nonzero potential cost savings or PO error cost, and the smallest unit of money, corresponding to one cent. We find that there are no substantial differences in the regression results when a number less than 0.01 is added. To capture the effect of the variation of SKU demand on the potential cost savings or PO error cost, we use the SKU unit time demand coefficient of variation (defined as  $\sigma/\mu$ ) instead of the standard deviation  $\sigma$  since the latter is highly positively correlated with the demand mean  $\mu$ . To capture the effect of the backlogging penalty cost on the potential cost savings or PO error cost, we use the backlogging penalty ratio (defined as  $\pi/h$ ) instead of the backlogging penalty cost per unit per unit time  $\pi$  since the latter is highly positively correlated with the holding cost per unit per unit time  $h$ .

We use the calculated potential cost savings or PO error cost and the related parameters for the selected SKUs as described for the previous numerical experiments to fit the predictive models. Table 9 shows the regression results for the potential cost savings predictive model. The R-square for this fit is 0.9675. Table 10 shows the regression results for the PO error cost predictive model. The R-square for this fit is 0.9717. According to the signs of the regression coefficients, both the potential cost savings and PO error cost decrease in the ordering cost  $K$ , and increase in the holding cost  $h$  and the backlogging penalty cost  $\pi$  as can be shown for the deterministic inventory model

<sup>12</sup>Note that the remaining PO error related parameters  $\beta$  and  $b_1$  are constant across all POs and hence all SKUs. The details can be found in Section 4.2

and can be observed for the stochastic one. Both the potential cost savings and PO error cost increase in PO error related parameters  $b_0$  and  $\alpha$ , since the greater these parameters, the greater the expected error size of a PO. Both the potential cost savings and PO error cost increase in the unit time demand mean  $\mu$ , as can be shown for the deterministic inventory model. Both the potential cost savings and PO error cost increase in the unit time demand coefficient of variation  $\sigma/\mu$ , since the more varied the demand, the larger the system inventory costs. Both the potential cost savings and PO error cost decrease in the lead time from the vendor to the retailer  $L$ , since the effect of PO errors is to increase lead time and lead time variability, and the larger the lead time without rework  $L$  is, the smaller this effect is relative to the lead time  $L$ .

Table 9: Potential Cost Savings Regression Results ( $n = 1,076,320$ )

	Estimate	Standard Error	$t$ value	$p$ value
Intercept	-3.12	$3.65e^{-3}$	-854.42	$\ll 0.001$
$\log(K)$	-0.24	$3.90e^{-4}$	-625.28	$\ll 0.001$
$\log(h)$	1.10	$6.64e^{-4}$	1650.66	$\ll 0.001$
$\log(\pi/h)$	0.56	$4.12e^{-4}$	1366.88	$\ll 0.001$
$b_0$	0.05	$4.22e^{-4}$	108.11	$\ll 0.001$
$\alpha$	0.12	$4.66e^{-4}$	259.81	$\ll 0.001$
$\log(a)$	0.07	$3.09e^{-4}$	214.66	$\ll 0.001$
$\log(\mu)$	1.14	$2.20e^{-4}$	5165.34	$\ll 0.001$
$\sigma/\mu$	0.01	$6.54e^{-4}$	12.28	$\ll 0.001$
$\log(L)$	-0.48	$4.59e^{-4}$	-1042.96	$\ll 0.001$

Table 10: PO Error Cost Regression Results ( $n = 1,076,320$ )

	Estimate	Standard Error	$t$ value	$p$ value
Intercept	-0.66	$3.09e^{-3}$	-214.76	$\ll 0.001$
$\log(K)$	-0.14	$3.30e^{-4}$	-426.04	$\ll 0.001$
$\log(h)$	0.68	$5.62e^{-4}$	1207.56	$\ll 0.001$
$\log(\pi/h)$	0.16	$3.49e^{-4}$	470.14	$\ll 0.001$
$b_0$	0.39	$3.57e^{-4}$	1091.06	$\ll 0.001$
$\alpha$	0.57	$3.94e^{-4}$	1440.06	$\ll 0.001$
$\log(a)$	0.45	$2.61e^{-4}$	1731.49	$\ll 0.001$
$\log(\mu)$	0.95	$1.86e^{-4}$	5128.42	$\ll 0.001$
$\sigma/\mu$	0.70	$5.53e^{-4}$	1259.90	$\ll 0.001$
$\log(L)$	-0.28	$3.88e^{-4}$	-725.90	$\ll 0.001$

We now consider the performance of the potential cost savings predictive model. The absolute error (defined as the absolute difference between the predicted value and the actual value) is on average \$0.26 with a median of \$0.03. The absolute relative error (defined as the average absolute error divided by the actual value) among SKUs with nonzero potential cost savings is on average

19.95% with a median of 12.69%. We can rank SKUs according to their actual or predicted cost savings from the highest to the lowest. The correlation between the actual rank and predicted rank is 0.99. The average absolute error between the predicted rank and actual rank is 28,690, which is 2.67% of the total number of SKUs to rank. The total number of SKUs to rank is 1,076,320. The largest absolute error between the predicted rank and actual rank is 537,440, which is 49.93% of the total number of SKUs to rank.

We now consider the performance of the PO error cost predictive model. The absolute error is on average \$1.09 with a median of \$0.17. The absolute relative error among SKUs with nonzero PO error cost is on average 17.65% with a median of 14.07%. We can rank SKUs according to their actual or predicted PO error cost from the highest to the lowest. The correlation between the actual rank and predicted rank is 0.98. The average absolute error between the predicted rank and actual rank is 37,160, which is 3.46% of the total number of SKUs to rank. The total number of SKUs to rank is 1,076,320. The largest absolute error between the predicted rank and actual rank is 511,100, which is 47.48% of the total number of SKUs to rank.

A retailer can use the predictive models for potential cost savings or PO error cost to select desired SKUs using only readily available information. We propose the following procedure for this purpose:

- Step 1: Estimate PO errors and rework related parameters (including  $b_0$ ,  $\alpha$ , and  $a$ ) for each vendor-department combination as in Section 4.
- Step 2: Estimate unit time demand  $\mu$ , standard deviation  $\sigma$ , and lead time from vendors to the retail DC  $L$  for each SKU.
- Step 3: Summarize related information for each SKU, including its inventory cost parameters ( $K$ ,  $h$ , and  $\pi$ ), error and rework related parameters (including  $b_0$ ,  $\alpha$ , and  $a$ ), unit time demand mean  $\mu$  and standard deviation  $\sigma$ , and lead time  $L$ .
- Step 4: Predict potential cost savings or PO error cost of every SKU using the predictive models.
- Step 5: Rank SKUs using the predicted cost savings or PO error cost, and choose SKUs with higher ranks as desired.

For this procedure to work for a general retailer, we have implicitly assumed that other retailers have the same regression coefficients for the potential cost savings or PO error cost predictive model as the retailer under study. The above procedure provides retailers with a way to select SKUs for adjusting and fine tuning existing inventory models or for reducing error size with readily available information about SKUs. The assumption is not very restrictive for retailers similar to our partner retailer, the details of which can be found in Section 4. The only fixed inputs from our partner retailer for the above regression analysis include the PO error related parameters  $b_1$  and  $\beta$ , which are constant over all SKUs and will not influence the ranking of SKUs. Even though the absolute value predictions may be off for these retailers, the ranking of SKUs should not be

influenced much by using these fixed inputs from our partner retailer. The above procedure may not be very suitable for retailers quite different from our partner retailer. However, the previous regression analysis still provides valuable insights in terms of qualitative guidance. For example, a retailer should focus on SKUs with shorter lead time to address PO errors.

Sometimes, a retailer may be interested in identifying problematic vendors with higher PO error cost so that it can collaborate with them to reduce the incidence of such errors. Our procedure can also be used to predict PO error cost at the vendor level by summing the predicted PO error cost for SKUs from each vendor. We have used our procedure to predict PO error cost at the vendor level when the fixed ordering cost is 100 dollars per order, the holding cost is 20% of the purchase cost and the backlogging penalty cost is 10 times of the holding cost. The absolute relative error is on average 15.04% with a median of 12.05%. We can rank vendors according to their actual or predicted PO error cost from the highest to the lowest. The correlation between the actual rank and predicted rank is 0.98. The average absolute error between the predicted rank and actual rank is 3.5, which is 3.30% of the total number of vendors to rank. The total number of vendors to rank is 106. The largest absolute error between the predicted rank and actual rank is 27, which is 25.47% of the total number of vendors to rank.

## 6 Conclusions

Vendor noncompliance problems are very common in retail practice. Both retailers and their vendors are interested in quantifying the impact of such problems on retail inventory management. Vendor noncompliance problems result in PO errors. Only one type of these PO errors, “quantity errors”, has been studied in the random yield literature, while many other types like “ticket errors” have received little attention by previous researchers. To the best of our knowledge, our paper is the first one to identify the other types of PO errors, and to perform empirical analysis on real retail data to obtain properties of the other PO errors. We find that the PO errors under study vary with order quantity in a way different from what is commonly assumed in the random yield literature. In order to analyze the influence of PO errors on a retailer’s ordering policy and inventory system cost, we propose a deterministic inventory model and a stochastic inventory model that account for these PO errors. We obtain several findings through analytically studying the deterministic inventory model, then numerically analyzing the stochastic inventory model with parameters estimated from real retail data. First, PO errors cost the retailer tens of dollars per SKU per year, and whether the retailer undercharges or overcharges its vendors for these PO errors depends on its inventory cost parameters. The PO error cost is higher for retailers with lower fixed ordering cost, higher holding cost, and higher backlogging penalty cost. The parties have fiercely debated over whether the chargebacks imposed by retailers on their vendors for these problems are too low or too high. Our findings provide valuable insights for the unsettled debate. Second, when accounting for PO errors, the retailer’s order size is almost always greater than if PO errors are not accounted for.

Third, the retailer can achieve considerable savings by adjusting its inventory policy to account for PO errors. The cost savings is higher for retailers with lower fixed ordering costs, higher holding cost, and higher backlogging penalty costs. In practice, a retailer may need to manage hundreds of thousands of SKUs and may purchase inventory management software from an outside vendor. We propose procedures and provide qualitative guidance using readily available SKU information for a retailer to easily identify SKUs for which adjusting and fine-tuning inventory system or reducing error size would be particularly beneficial. With such procedures, a retailer does not need to pay its software vendor for every SKU it owns in order to identify the ones that suffer the most from PO errors.

There are several possible directions for future research. First, one might want to relax the assumption that a PO can only be used after it finishes rework. For some types of errors, the correct portion of a PO can be used before it finishes rework. Secondly, one can study the joint replenishment problem for two SKUs while accounting for PO errors. One new feature in the two SKU case is that PO errors depend on the number of SKUs in a PO. Substitution errors, under which a retailer orders SKU A but receives SKU B, can also be accounted for in the two SKU model. Third, it would be interesting to find ways to deal with uncorrectable errors that are not covered in this paper. For example, time error cannot be corrected. A PO with a time error either arrives earlier or later than stipulated in the contract. Finally, future work might want to examine ways to prevent and reduce PO errors from the source. The reduction of PO errors might be achieved through retailers providing the right incentives to their vendors, such as contracts addressing PO quality.

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## Appendix A

*Proof of Theorem 1:*

For any  $Q \leq Q_s$ ,  $C'_r(Q) = \frac{-\lambda K}{Q^2} + \frac{h\pi}{2(h+\pi)} + c_r a \lambda \rho'(Q) \leq c_r a \lambda \rho'(Q) \leq 0$ . The first inequality is true because  $\frac{-\lambda K}{Q^2}$  decreases in  $Q$ , and  $\frac{-\lambda K}{Q_s^2} + \frac{h\pi}{2(h+\pi)} = 0$ . The second inequality is true because  $\rho(Q)$  decreases in  $Q$ . Hence,  $C_r(Q)$  decreases in  $Q$  when  $Q \leq Q_s$ . Since  $Q_r$  minimizes  $C_r(Q)$ , we must have  $Q_r \geq Q_s$ .  $\square$

*Proof of Theorem 2:*

We know that  $C_s(Q_s) = \sqrt{\frac{2\lambda K h \pi}{h + \pi}}$ . Hence,  $\frac{dC_s(Q_s)}{dK} = \sqrt{\frac{\lambda h \pi}{2K(h + \pi)}}$ .  $C_r(Q_r) = \frac{\lambda K}{Q_r} + \frac{1}{2} \frac{h\pi}{h + \pi} Q_r +$

$c_r a \lambda \rho(Q_r)$ . Therefore,  $\frac{dC_r(Q_r)}{dK} = \frac{\lambda}{Q_r}$ , according to the Envelope Theorem. Hence,  $\frac{C_r(Q_r)}{dK} - \frac{C_s(Q_s)}{dK} = \frac{\lambda}{Q_r} - \sqrt{\frac{\lambda h \pi}{2K(h + \pi)}} \leq 0$ . The inequality is due to the fact that  $Q_r \geq Q_s = \sqrt{\frac{2\lambda K(h + \pi)}{h\pi}}$ . Hence,  $C_r(Q_r) - C_s(Q_s)$  decreases in the fixed ordering cost  $K$ .  $\square$

*Proof of Theorem 3:*

Now we have  $\rho(Q) = cQ^{\beta-1}$ . Hence  $C_r(Q_s) = \sqrt{\frac{2\lambda K h \pi}{h + \pi}} + c_r a c \lambda Q_s^{\beta-1}$ , where  $Q_s = \sqrt{\frac{2\lambda K(h + \pi)}{h\pi}}$ .  $\frac{dC_r(Q_s)}{dK} = \sqrt{\frac{\lambda h \pi}{2K(h + \pi)}} - c_r a c \lambda (1 - \beta) Q_s^{\beta-2} \sqrt{\frac{\lambda(h + \pi)}{2K h \pi}}$ .  $C_r(Q_r) = \frac{\lambda K}{Q_r} + \frac{1}{2} \frac{h\pi}{h + \pi} Q_r + c_r a c \lambda Q_r^{\beta-1}$ .

According to the Envelope Theorem,  $\frac{dC_r(Q_r)}{dK} = \frac{\lambda}{Q_r}$ .  $\frac{dC_r(Q_s)}{dK} - \frac{dC_r(Q_r)}{dK} = \sqrt{\frac{\lambda h \pi}{2K(h + \pi)}} - c_r a c \lambda (1 - \beta) Q_s^{\beta-2} \sqrt{\frac{\lambda(h + \pi)}{2K h \pi}} - \frac{\lambda}{Q_r} \Rightarrow \frac{dC_r(Q_s)}{dK} - \frac{dC_r(Q_r)}{dK} = \sqrt{\frac{\lambda h \pi}{2K(h + \pi)}} - \frac{\lambda}{Q_r} - c_r a c \lambda (1 - \beta) Q_s^{\beta-2} \sqrt{\frac{\lambda(h + \pi)}{2K h \pi}} = c_r a \lambda \sqrt{\frac{\lambda(h + \pi)}{2K h \pi}} \left[ \frac{h\pi}{h + \pi} \left(1 - \frac{Q_s}{Q_r}\right) - c(1 - \beta) Q_s^{\beta-2} \right]$ . Let  $f(K) = \frac{h\pi}{h + \pi} \left(1 - \frac{Q_s}{Q_r}\right) - c(1 - \beta) Q_s^{\beta-2}$ . Then,  $\frac{dC_r(Q_s)}{dK} - \frac{dC_r(Q_r)}{dK} = c_r a \lambda \sqrt{\frac{\lambda(h + \pi)}{2K h \pi}} f(K)$ .

Now, we want to show that  $\frac{dC_r(Q_s)}{dK} - \frac{dC_r(Q_r)}{dK} < 0$ . First of all, as  $K \rightarrow \infty$ ,  $f(K) \rightarrow 0$  since  $Q_s \rightarrow Q_r$  and  $Q_s \rightarrow \infty$ . Furthermore,  $\frac{\lambda(h + \pi)}{2K h \pi} \rightarrow 0$  as  $K \rightarrow \infty$ . Hence,  $\frac{dC_r(Q_s)}{dK} - \frac{dC_r(Q_r)}{dK} \rightarrow 0$  as  $K \rightarrow \infty$ . Second, we will show that the derivative of  $\frac{dC_r(Q_s)}{dK} - \frac{dC_r(Q_r)}{dK}$  with regard to  $K$  is greater than 0 by showing that  $\frac{df(K)}{dK} > 0$ . We can derive that  $\frac{df(K)}{dK} =$

$-\frac{h\pi}{h + \pi} \left( Q_r \frac{dQ_s}{dK} - Q_s \frac{dQ_r}{dK} \right) \frac{1}{Q_r^2 c_r a \lambda} + c(1 - \beta)(2 - \beta) Q_s^{\beta-3} \frac{dQ_s}{dK}$ . We know that  $-\frac{\lambda K}{Q_s^2} + \frac{1}{2} \frac{h\pi}{h + \pi} = 0$ . To calculate  $\frac{dQ_s}{dK}$ , take the derivative with regard to  $K$  on both sides. We get  $\frac{2\lambda K}{Q_s^3} \frac{dQ_s}{dK} - \frac{\lambda}{Q_s^2} = 0$ .

$\Rightarrow \frac{dQ_s}{dK} = \frac{Q_s}{2K}$ . We also know that  $-\frac{\lambda K}{Q_r^2} + \frac{1}{2} \frac{h\pi}{h + \pi} - c_r a c \lambda (1 - \beta) Q_r^{\beta-2} = 0$ . To calculate  $\frac{dQ_r}{dK}$ , take the derivative with regard to  $K$  on both sides. We get  $\frac{2\lambda K}{Q_r^3} \frac{dQ_r}{dK} - \frac{\lambda}{Q_r^2} + c_r a c \lambda (1 - \beta)(2 - \beta) Q_r^{\beta-3} \frac{dQ_r}{dK} = 0 \Rightarrow \frac{dQ_r}{dK} = \frac{Q_r}{2K + c_r a c (1 - \beta)(2 - \beta) Q_r^\beta}$ . Substituting  $\frac{dQ_s}{dK}$

and  $\frac{dQ_r}{dK}$  into  $\frac{df(K)}{dK}$ , we get that  $\frac{df(K)}{dK} = -\frac{h\pi}{h + \pi} \left( \frac{Q_r Q_s}{2K} - \frac{Q_r Q_s}{2K + c_r a c (1 - \beta)(2 - \beta) Q_r^\beta} \right) \frac{1}{Q_r^2 c_r a \lambda} + c(1 - \beta)(2 - \beta) Q_s^{\beta-3} \frac{Q_s}{2K} \Rightarrow \frac{df(K)}{dK} = \frac{Q_s Q_r^{\beta-1} c(1 - \beta)(2 - \beta)}{2K(2K + c_r a c (1 - \beta)(2 - \beta) Q_r^\beta)} g(K)$ , where  $g(K)$  is defined as

$-\frac{h\pi}{\lambda(h + \pi)} + \frac{2K(2K + c_r a c (1 - \beta)(2 - \beta) Q_r^\beta)}{2K} Q_r^{1-\beta} Q_s^{\beta-3}$ . We know that  $g(K) > -\frac{h\pi}{\lambda(h + \pi)} + \frac{2K}{2K} Q_r^{1-\beta} Q_s^{\beta-3} \geq -\frac{h\pi}{\lambda(h + \pi)} + 2K Q_s^{-2} = -\frac{h\pi}{\lambda(h + \pi)} + 2K \sqrt{\frac{h\pi}{2K\lambda(h + \pi)}} = 0$ . The first in-

equality is due to  $c_r ac(1-\beta)(2-\beta)Q_r^\beta > 0$  because  $0 < \beta < 1$  and  $c > 0$ . The second inequality is due to  $Q_r \geq Q_s$ . Hence,  $\frac{df(K)}{dK} = \frac{Q_s Q_r^{\beta-1} c(1-\beta)(2-\beta)}{2K(2K + c_r ac(1-\beta)(2-\beta)Q_r^\beta)} g(K) > 0$ . As a result, the derivative of  $\frac{dC_r(Q_s)}{dK} - \frac{dC_r(Q_r)}{dK}$  with regard to  $K$  is greater than 0. We have shown previously that  $\frac{dC_r(Q_s)}{dK} - \frac{dC_r(Q_r)}{dK} \rightarrow 0$  as  $K \rightarrow \infty$ . Therefore,  $\frac{dC_r(Q_s)}{dK} - \frac{dC_r(Q_r)}{dK} < 0$ . Hence, the cost savings  $C_r(Q_s) - C_r(Q_r)$  decreases in the fixed ordering cost  $K$ .  $\square$

*Proof of Theorem 4:*

We know that  $C_s(Q_s) = \sqrt{\frac{2\lambda K h \pi}{h + \pi}}$ . Hence,  $\frac{dC_s(Q_s)}{d\pi} = \frac{1}{2} \sqrt{2K\lambda} \frac{h + \pi}{h\pi} \frac{h^2}{(h + \pi)^2} = \frac{Q_s}{2} \frac{h^2}{(h + \pi)^2}$ . The second equation is due to the fact that  $Q_s = \sqrt{\frac{2K\lambda(h + \pi)}{h\pi}}$ .  $C_r(Q_r) = \frac{\lambda K}{Q_r} + \frac{1}{2} \frac{h\pi}{h + \pi} Q_r + c_r a \lambda \rho(Q_r)$ . Therefore,  $\frac{dC_r(Q_r)}{d\pi} = \frac{Q_r}{2} \frac{h^2}{(h + \pi)^2}$  according to the Envelope Theorem. Hence,  $\frac{dC_r(Q_r)}{d\pi} - \frac{dC_s(Q_s)}{d\pi} = \frac{Q_r}{2} \frac{h^2}{(h + \pi)^2} - \frac{Q_s}{2} \frac{h^2}{(h + \pi)^2} \geq 0$  since  $Q_r \geq Q_s$ . Hence  $C_r(Q_r) - C_s(Q_s)$  increases in the backlogging penalty cost  $\pi$ . The property that PO error cost  $C_r(Q_r) - C_s(Q_s)$  increases in the unit holding cost can be proved similarly with  $\pi$  replaced by  $h$  in the previous derivation.  $\square$

*Proof of Theorem 5:*

Now we have  $\rho(Q) = cQ^{\beta-1}$ . Therefore,  $C_r(Q_s) = \sqrt{\frac{2\lambda K h \pi}{h + \pi}} + c_r ac \lambda Q_s^{\beta-1}$ , where  $Q_s = \sqrt{\frac{2K\lambda(h + \pi)}{h\pi}}$ . Hence,  $\frac{dC_r(Q_s)}{d\pi} = \frac{Q_s}{2} \frac{h^2}{(h + \pi)^2} - c_r ac \lambda (1-\beta) Q_s^{\beta-2} \frac{dQ_s}{d\pi} = \frac{Q_s}{2} \frac{h^2}{(h + \pi)^2} + c_r ac \lambda (1-\beta) Q_s^{\beta-1} \frac{Q_s}{2} \frac{h}{h + \pi}$ , since  $\frac{dQ_s}{d\pi} = -\frac{Q_s}{2} \frac{h}{\pi(h + \pi)}$ .  $C_r(Q_r) = \frac{\lambda K}{Q_r} + \frac{1}{2} \frac{h\pi}{h + \pi} Q_r + c_r ac \lambda Q_r^{\beta-1}$ . According to the Envelope Theorem,  $\frac{dC_r(Q_r)}{d\pi} = \frac{Q_r}{2} \frac{h^2}{(h + \pi)^2}$ .  $\frac{dC_r(Q_s)}{d\pi} - \frac{dC_r(Q_r)}{d\pi} = \frac{Q_s}{2} \frac{h^2}{(h + \pi)^2} + c_r ac \lambda (1-\beta) Q_s^{\beta-1} \frac{Q_s}{2} \frac{h}{h + \pi} - \frac{Q_r}{2} \frac{h^2}{(h + \pi)^2} = \frac{h}{2(h + \pi)} \left( \frac{Q_s h}{h + \pi} - \frac{Q_r h}{h + \pi} + c_r a \lambda c(1-\beta) Q_s^{\beta-1} \frac{1}{\pi} \right)$ . Let  $f(\pi) = \frac{Q_s h}{h + \pi} - \frac{Q_r h}{h + \pi} + c_r a \lambda c(1-\beta) Q_s^{\beta-1} \frac{1}{\pi}$ . Therefore,  $\frac{dC_r(Q_s)}{d\pi} - \frac{dC_r(Q_r)}{d\pi} = \frac{h}{2(h + \pi)} f(\pi)$ .

Now we want to show that  $\frac{dC_r(Q_s)}{d\pi} - \frac{dC_r(Q_r)}{d\pi} > 0$ . First of all, as  $\pi \rightarrow \infty$ ,  $f(\pi) \rightarrow 0$  since  $Q_s \rightarrow \sqrt{\frac{2K\lambda(h + \pi)}{h\pi}}$  and  $Q_r \rightarrow \bar{Q}_r$ , where  $\frac{h}{2} - \frac{K\lambda}{\bar{Q}_r^2} - c_r ac \lambda (1-\beta) \bar{Q}_r^{\beta-2} = 0$ . Second, we will show that the derivative of  $\frac{dC_r(Q_s)}{d\pi} - \frac{dC_r(Q_r)}{d\pi}$  with regard to  $\pi$  is smaller than 0 by showing that  $\frac{df(\pi)}{d\pi} > 0$ . We can derive that  $\frac{df(\pi)}{d\pi} = \frac{dQ_s}{d\pi} \frac{h}{h + \pi} + Q_s \frac{h^2}{(h + \pi)^2} - \frac{dQ_r}{d\pi} \frac{h}{h + \pi} - Q_r \frac{h^2}{(h + \pi)^2} + c_r a \lambda c(1-\beta)(\beta-1) Q_s^{\beta-2} \frac{1}{\pi} - c_r a \lambda c(1-\beta) Q_s^{\beta-1} \frac{1}{\pi^2}$ . We know that  $-\frac{\lambda K}{Q_s^2} + \frac{1}{2} \frac{h}{h + \pi} = 0$ . To calculate  $\frac{dQ_s}{d\pi}$ , take the derivative with regard to  $\pi$  on both sides. We get  $\frac{2\lambda K}{Q_s^3} \frac{dQ_s}{d\pi} + \frac{1}{2} \frac{h\pi}{(h + \pi)^2} = 0$

$\Rightarrow \frac{dQ_s}{d\pi} = -\frac{1}{4} \frac{h\pi}{(h+\pi)^2} \frac{Q_s^3}{2K\lambda}$ . We also know that  $-\frac{\lambda K}{Q_r^2} + \frac{1}{2} \frac{h}{h+\pi} - c_r a c \lambda (1-\beta) Q_r^{\beta-2} = 0$ . To calculate  $\frac{dQ_r}{d\pi}$ , take the derivative with regard to  $\pi$  on both sides. We get  $\frac{2\lambda K}{Q_r^3} \frac{dQ_r}{d\pi} + \frac{1}{2} \frac{h\pi}{(h+\pi)^2} + \frac{1}{2} \frac{h\pi}{(h+\pi)^2} c_r a c \lambda (1-\beta)(2-\beta) Q_r^{\beta-3} \frac{dQ_r}{d\pi} = 0 \Rightarrow \frac{dQ_r}{d\pi} = \frac{-\frac{1}{2} \frac{h\pi}{(h+\pi)^2}}{2 \frac{K\lambda}{Q_r^3} + c_r a c (1-\beta)(2-\beta) Q_r^{\beta-3} \lambda}$ . The denominator  $2 \frac{K\lambda}{Q_r^3} + c_r a c (1-\beta)(2-\beta) Q_r^{\beta-3} \lambda > 2 \frac{K\lambda}{Q_r^3} > 2 \frac{K\lambda}{Q_s^3}$ . The second inequality holds because  $Q_r > Q_s$ . Hence,  $\frac{dQ_r}{d\pi} > \frac{dQ_s}{d\pi}$ . Therefore,  $\frac{df(\pi)}{d\pi} > 0$ . As a result, the derivative of  $\frac{dC_r(Q_s)}{d\pi} - \frac{dC_r(Q_r)}{d\pi}$  with regard to  $\pi$  is less than 0. We have shown previously that  $\frac{dC_r(Q_s)}{d\pi} - \frac{dC_r(Q_r)}{d\pi} \rightarrow 0$  as  $\pi \rightarrow \infty$ . Therefore,  $\frac{dC_r(Q_s)}{d\pi} - \frac{dC_r(Q_r)}{d\pi} > 0$ . Hence, the cost savings  $C_r(Q_s) - C_r(Q_r)$  increases in the backlogging penalty cost  $\pi$ . The property that cost savings  $C_r(Q_s) - C_r(Q_r)$  increases in the unit holding cost can be proved similarly with  $\pi$  replaced by  $h$  in the previous derivation.  $\square$

*Proof of Theorem 6:*

First, we will show that the PO error cost  $C_r(Q_r) - C_s(Q_s)$  increases in the demand rate  $\lambda$ . We know that  $C_s(Q_s) = \sqrt{\frac{2\lambda K h \pi}{h+\pi}}$ . Hence,  $\frac{dC_s(Q_s)}{d\lambda} = \sqrt{\frac{2K h \pi}{h+\pi}} \frac{1}{2\sqrt{\lambda}}$ . We also know that  $C_r(Q_r) = \frac{\lambda K}{Q_r} + \frac{1}{2} \frac{h\pi}{h+\pi} Q_r + c_r a \lambda \rho(Q_r)$ . Therefore,  $\frac{dC_r(Q_r)}{d\lambda} = \frac{K}{Q_r} + c_r a \rho(Q)$ , according to the Envelope Theorem. Hence,  $\frac{dC_r(Q_r)}{d\lambda} - \frac{dC_s(Q_s)}{d\lambda} = \frac{K}{Q_r} + c_r a \rho(Q) - \sqrt{\frac{2K h \pi}{h+\pi}} \frac{1}{2\sqrt{\lambda}}$ . Since  $Q_r$  is the minimizer for  $C_r(Q)$ , we have  $\frac{1}{2} \frac{h\pi}{h+\pi} - \frac{K\lambda}{Q_r^2} + c_r a \rho'(Q_r) \lambda = 0$ . Since  $\rho(Q) = cQ^{\beta-1}$ , we have  $\frac{1}{2} \frac{h\pi}{h+\pi} Q_r - \frac{K\lambda}{Q_r} - (1-\beta) c_r a \rho(Q_r) \lambda = 0$ . Hence,  $\frac{K\lambda}{Q_r} + c_r a \rho(Q_r) \lambda > \frac{K\lambda}{Q_r} + (1-\beta) c_r a \rho(Q_r) \lambda = \frac{1}{2} \frac{h\pi}{h+\pi} Q_r > \sqrt{\frac{2K h \pi}{h+\pi}} \frac{1}{2\sqrt{\lambda}}$ . The first inequality is because  $\beta < 1$ , and the second one is because  $Q_r > Q_s = \sqrt{\frac{2K\lambda(h+\pi)}{h\pi}}$ . Hence,  $\frac{dC_r(Q_r)}{d\lambda} - \frac{dC_s(Q_s)}{d\lambda} > 0$ . That is,  $C_r(Q_r) - C_s(Q_s)$  increases in the demand rate  $\lambda$ .

Next, we will show that the cost savings  $C_r(Q_s) - C_r(Q_r)$  increases in the demand rate  $\lambda$ . We know that  $C_r(Q_s) = \sqrt{\frac{2\lambda K h \pi}{h+\pi}} + c_r a \rho(Q_s) \lambda$ . Hence,  $\frac{dC_r(Q_s)}{d\lambda} = \sqrt{\frac{2K h \pi}{h+\pi}} \frac{1}{2\sqrt{\lambda}} + c_r a \rho(Q_s) + c_r a \lambda \rho'(Q_s) \frac{dQ_s}{d\lambda} = \sqrt{\frac{2K h \pi}{h+\pi}} \frac{1}{2\sqrt{\lambda}} + \frac{\beta+1}{2} c_r a \rho(Q_s)$  since  $\frac{dQ_s}{d\lambda} = \frac{Q_s}{2\lambda}$  and  $\rho(Q) = cQ^{\beta-1}$ . Hence,  $\frac{dC_r(Q_s)}{d\lambda} - \frac{dC_r(Q_r)}{d\lambda} = \sqrt{\frac{2K h \pi}{h+\pi}} \frac{1}{2\sqrt{\lambda}} + \frac{\beta+1}{2} c_r a \rho(Q_s) - \frac{K}{Q_r} - c_r a \rho(Q) = \frac{K}{Q_s} + \frac{\beta+1}{2} c_r a \rho(Q_s) - \frac{K}{Q_r} - c_r a \rho(Q)$ . Since  $Q_r$  is the minimizer for  $C_r(Q)$ , we have

$$\frac{1}{2} \frac{h\pi}{h+\pi} - \frac{K\lambda}{Q_r^2} + c_r a \rho'(Q_r) \lambda = 0 \Rightarrow$$

$$\frac{1}{2} \frac{h\pi}{h+\pi} Q_r - \frac{K\lambda}{Q_r} - (1-\beta)c_r a \rho(Q_r) \lambda = 0 \quad (10)$$

Since  $Q_s$  is the minimizer for  $C_s(Q)$ , we have

$$\begin{aligned} \frac{1}{2} \frac{h\pi}{h+\pi} - \frac{K\lambda}{Q_s^2} &= 0 \Rightarrow \\ \frac{1}{2} \frac{h\pi}{h+\pi} Q_s - \frac{K\lambda}{Q_s} &= 0 \end{aligned} \quad (11)$$

Subtracting equation (11) from equation (10), we get  $\frac{K\lambda}{Q_s} - \frac{K\lambda}{Q_r} = \frac{1}{2\lambda} \frac{h\pi(Q_r - Q_s)}{h+\pi} + (1-\beta)c_r a \rho(Q_r)$ . Hence,  $\frac{dC_r(Q_s)}{d\lambda} - \frac{dC_r(Q_r)}{d\lambda} = \frac{\beta+1}{2} c_r a \rho(Q_s) + \frac{K}{Q_s} - \frac{K}{Q_r} - c_r a \rho(Q_r) = \frac{1}{2\lambda} \frac{h\pi(Q_r - Q_s)}{h+\pi} + \frac{\beta+1}{2} c_r a \rho(Q_s) - \beta c_r a \rho(Q_r) \geq 0$  since  $Q_r \geq Q_s$ ,  $\beta < 1$  and  $\rho(Q_s) \geq \rho(Q_r)$ . Therefore, the cost savings  $C_r(Q_s) - C_r(Q_r)$  increases in the demand rate  $\lambda$ .  $\square$

*Proof of Proposition 7:*

We know that  $E(\tilde{W}(Q))/Q = cp(Q)Q^{\beta-1}$ , where  $c$  is defined as  $e^{(\alpha+\sigma_s^2/2)}$ . The probability that there is an error  $p(Q) = \frac{e^{b_0+b_1 \log Q}}{1+e^{b_0+b_1 \log Q}} = \frac{dQ^{b_1}}{1+dQ^{b_1}}$ , where  $d$  is defined as  $e^{b_0}$ . And  $\frac{dp(Q)}{dQ} = \frac{db_1 Q^{b_1-1}}{(1+dQ^{b_1})^2} = p(Q) \frac{b_1}{Q(1+dQ^{b_1})}$ . We have

$$\begin{aligned} \frac{dE(\tilde{W}(Q))/Q}{dQ} &= (\beta-1)cp(Q)Q^{\beta-2} + cQ^{\beta-1} \frac{dp(Q)}{dQ} \\ &= (\beta-1)cp(Q)Q^{\beta-2} + cQ^{\beta-1} p(Q) \frac{b_1}{Q(1+dQ^{b_1})} \\ &= cp(Q)Q^{\beta-2} \left( (\beta-1) + \frac{b_1}{(1+dQ^{b_1})} \right) \\ &\leq cp(Q)Q^{\beta-2} ((\beta-1) + b_1) \end{aligned}$$

The last inequality is true because  $dQ^{b_1} \geq 0$  for any  $Q \geq 0$ . Hence, if  $\beta + b_1 \leq 1$ , then  $\frac{dE(\tilde{W}(Q))/Q}{dQ} \leq 0$ . That is,  $E(\tilde{W}(Q))/Q$  decreases in  $Q$ .  $\square$

## Appendix B

We will first derive the mean of the error size for a PO of a single SKU with order quantity  $Q$  given there is an error. We know that given  $\tilde{W}(Q) > 0$ ,  $\log(\tilde{W}(Q)) = \alpha + \beta \log(Q) + \gamma \log(n) + e_s$  with the number of SKUs  $n$  equal to 1, and  $e_s$  normally distributed with mean 0 and variance  $\sigma_s^2$ . Hence, given  $\tilde{W}(Q) > 0$ , we have  $\tilde{W}(Q) = e^\alpha Q^\beta \exp(e_s)$ , where  $\exp(e_s)$  has a log-normal distribution with mean  $e^{\sigma_s^2/2}$ . Therefore,  $E(\tilde{W}(Q)|\tilde{W}(Q) > 0) = e^{\alpha+\sigma_s^2/2} Q^\beta$ . In addition,

$$E(\tilde{W}(Q)) = E(\tilde{W}(Q)|\tilde{W}(Q) > 0)P(\tilde{W}(Q) > 0) = p(Q)e^{\alpha+\sigma_s^2/2}Q^\beta.$$

Then we will derive the the rework time mean  $E(\tilde{t}_r(Q))$  and variance  $Var(\tilde{t}_r(Q))$ . Recall that  $e_s$  is normally distributed with mean 0 and variance  $\sigma_s^2$ , and  $exp(e_s)$  has a log-normal distribution with mean  $e^{\sigma_s^2/2}$  and variance  $(e^{\sigma_s^2} - 1)e^{\sigma_s^2}$ , hence

$$\begin{aligned} E(\tilde{t}_r(Q)) &= E(\tilde{t}_r(Q)|\tilde{W}(Q) > 0)P(\tilde{W}(Q) > 0) + E(\tilde{t}_r(Q)|\tilde{W}(Q) = 0)P(\tilde{W}(Q) = 0) \\ &= E(a\tilde{W}(Q) + e^r|\tilde{W}(Q) > 0)p(Q) + 0 \times P(\tilde{W}(Q) = 0) \\ &= aE(\tilde{W}(Q)|\tilde{W}(Q) > 0)p(Q) \\ &= ap(Q)e^{(\alpha+\sigma_s^2/2)}Q^\beta \end{aligned}$$

and,

$$\begin{aligned} Var(\tilde{t}_r(Q)) &= E(\tilde{t}_r(Q)^2) - E^2(\tilde{t}_r(Q)) \\ &= E(\tilde{t}_r(Q)^2|\tilde{W}(Q) > 0)P(\tilde{W}(Q) > 0) + E(\tilde{t}_r(Q)^2|\tilde{W}(Q) = 0)P(\tilde{W}(Q) = 0) - E^2(\tilde{t}_r(Q)) \\ &= (Var(\tilde{t}_r(Q)|\tilde{W}(Q) > 0) + E^2(\tilde{t}_r(Q)|\tilde{W}(Q) > 0))p(Q) + 0 \times P(\tilde{W}(Q) = 0) - E^2(\tilde{t}_r(Q)) \\ &= (Var(a\tilde{W}(Q) + e^r|\tilde{W}(Q) > 0) + E^2(a\tilde{W}(Q) + e^r|\tilde{W}(Q) > 0))p(Q) - E^2(\tilde{t}_r(Q)) \\ &= [a^2Var(\tilde{W}(Q)|\tilde{W}(Q) > 0) + \sigma_r^2 + (ae^{(\alpha+\sigma_s^2/2)}Q^\beta)^2]p(Q) - (ap(Q)e^{(\alpha+\sigma_s^2/2)}Q^\beta)^2 \\ &= [a^2Var(e^\alpha Q^\beta exp(e_s)) + \sigma_r^2 + (ae^{(\alpha+\sigma_s^2/2)}Q^\beta)^2]p(Q) - (ap(Q)e^{(\alpha+\sigma_s^2/2)}Q^\beta)^2 \\ &= [(ae^\alpha Q^\beta)^2(e^{\sigma_s^2} - 1)e^{\sigma_s^2} + \sigma_r^2 + (ae^{(\alpha+\sigma_s^2/2)}Q^\beta)^2]p(Q) - (ap(Q)e^{(\alpha+\sigma_s^2/2)}Q^\beta)^2 \\ &= a^2e^{2\alpha+\sigma_s^2}Q^{2\beta}(e^{\sigma_s^2} - p(Q))p(Q) + \sigma_r^2p(Q). \end{aligned}$$

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